*Journal of Statistical Physics, Vol. 102, Nos. 3/4, 2001*

# Let's Baxterise[1]

**S. Boukraa[2] and J.-M. Maillard[3]**

*Received April 10, 2000; final September 14, 2000*

We recall the concept of Baxterisation of an $R$-matrix, or of a monodromy matrix, which corresponds to build, from one point in the $R$-matrix parameter space, the algebraic variety where the spectral parameter(s) live. We show that the Baxterisation, which amounts to studying the iteration of a birational transformation, is a "win–win" strategy: it enables to discard efficiently the non-integrable situations, focusing directly on the two interesting cases where the algebraic varieties are of the so-called "general type" (finite order iteration) or are Abelian varieties (infinite order iteration). We emphasize the heuristic example of the sixteen vertex model and provide a complete description of the finite order iterations situations for the Baxter model. We show that the Baxterisation procedure can be introduced in much larger frameworks where the existence of some underlying Yang–Baxter structure is not used: we Baxterise L-operators, local quantum Lax matrices, and quantum Hamiltonians.

**KEY WORDS:** Baxterisation; Yang–Baxter equations; birational transformations; discrete dynamical systems; elliptic curves; lattice statistical mechanics; integrable mappings; L-operator; local quantum Lax matrices.

## 1. INTRODUCTION

The Yang–Baxter equations are known to be a sufficient condition[4] for the commutation of transfer matrices. Moreover, it has been shown that the commutation of transfer matrices necessarily yields a parameterization of the $R$-matrices in term of *algebraic varieties*,[(2)] and that the set of *inversion relations*, combined together with the geometric symmetries of the lattice,

---

[1] Dedicated to R. J. Baxter on the occasion of his 60th birthday.

[2] Université de Blida, Institut d'Aéronautique, Route de Soumaa, BP 270, Blida Algeria.

[3] LPTHE, Tour 16, 1er étage, 4 Place Jussieu, 75252 Paris Cedex, France; e-mail: maillard@lpthe.jussieu.fr

[4] And to some extent, necessary condition.[(1)]

yield a set of *birational* symmetries[5] of the parameter space of the model, which are *discrete symmetries* of the Yang–Baxter equations.(4, 5) Generically this set of birational symmetries is an infinite set. Combining these facts, one gets the following result: the Yang–Baxter integrability is *necessarily* parameterized in term of *algebraic varieties* having a set of *discrete birational symmetries*. An algebraic variety with an infinite set of (birational) symmetries cannot be an algebraic variety of the so-called "general type."(2) In the following this set of discrete birational symmetries will be mainly seen as generated by the iteration of a birational transformation, thus canonically associating a *discrete dynamical system*. This birational transformation can be of finite order, yielding a finite set of discrete symmetries: many Yang–Baxter integrable models correspond to this *finite order* situation(6) (see below: RSOS models,(7) integrable chiral Potts model,(11) free-fermion models, tetrahedron relations,(12–14)...). However, such birational transformations are generically (seen as a discrete dynamical system) *infinite order* transformations. The iteration of one point under such an infinite order birational transformation, yields an infinite number of points which can actually "densify" an algebraic variety (elliptic curve, Abelian surface,..., see refs. 15 and 16). One can thus deduce the algebraic variety from the iteration of one point by this birational transformation. This iteration procedure actually *solves*(4, 5) *the so-called "Baxterisation" problem* introduced by V. Jones in a framework of knot theory. The Baxterisation problem corresponds to actually find, from one (isolated) $R$-matrix, satisfying the Yang–Baxter equation, a whole family of $R$-matrices depending on one, or several, "spectral parameter(s)" satisfying the Yang–Baxter equation. In other words, the Baxterisation problem corresponds to actually build, from one point in the $R$-matrix parameter space (one Yang–Baxter integrable $R$-matrix), the algebraic variety where the spectral parameter(s) live. All this is also true for higher dimensional generalizations (tetrahedron relations,(13) ...) of the Yang–Baxter equations, and for Baxterisation problems in dimensions greater than two.(17)

This paper will be illustrated by many examples of Baxterisation, in particular, the heuristic example of the Baxterisation of the sixteen vertex model which is, generically, non Yang–Baxter integrable, with a detailed analysis of the finite order situations of the Baxter model. One will finally show that the Baxterisation procedure can be introduced in much larger

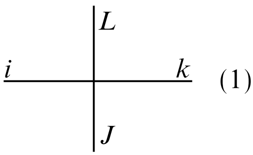
[5] A birational transformation of several (complex) variables is a rational transformation such that its inverse is *also* a rational transformation. Birational transformations play a crucial role in the study of algebraic varieties as illustrated by the Italian geometers of the last century. More than anyone else, the creation of the Italian school of projective and algebraic geometry is due to Cremona.(3)

frameworks where the existence of some underlying Yang–Baxter structure is not clear, and not used. For instance, we will give several examples of Baxterisation of quantum Hamiltonians. We will also give several examples of Baxterisation of differential operators, starting with the simple example of the Baxterisation of the Toda L-operator, and then giving examples of Baxterisation of other simple local quantum Lax matrices.

A large part of this paper corresponds to a review of previous works by the authors, but we also provide new results,[6] or we provide a new point of view: for instance, Section 7 revisit already known examples (the Baxter model, the t-J model, the Perk–Schultz model,...) but provides a new algebraic geometry point of view (the quantum hamiltonian limit is associated to the singular points of the parameter space, the equivalent of the base points of an elliptic foliation) and also underlies the analysis of the complexity of the iteration calculations (see Section 7.5). Therefore this paper is not organized as a review paper but, rather, as a self contained heuristic paper organized along the unifying concept of Baxterisation.

## 2. THE DISCRETE SYMMETRY GROUP ASSOCIATED WITH THE BAXTERISATION PROCEDURE

Let us consider a quite general vertex model where one direction, denoted as direction (1), is singled out. Pictorially this can be represented as follows:




(1)

where $i$ and $k$ (corresponding to direction (1)) can take $q$ values, while $J$ and $L$, in the other direction, take $m$ values.

One can define a "partial" transposition on direction (1) denoted $t_1$. The action of $t_1$ on the $R$-matrix is given by:[15, 16]

$$(t_1 R)^{iJ}_{kL} = R^{kJ}_{iL} \tag{2}$$

[6] In particular, the symmetric equation (48) is new, as well as its equivalence with the biquadratic (49). The results of Section 6, providing polynomial representations of the multiplication of the shift by an integer and the associated finite order conditions, are new: of course, the finite order conditions for the symmetric six vertex model, or the Baxter model (then associated with the various RSOS models see for instance, ref. 7) correspond to the "rational cases" already seen many years ago within the zero-field $XXZ$ or $XYZ$ Hamiltonian models.[8–10] Section 8 also provides simple new results.

The $R$-matrix is a $(qm) \times (qm)$ matrix which can be seen as $q^2$ blocks which are $m \times m$ matrices:

$$R = \begin{pmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,q] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,q] \\ A[3,1] & A[3,2] & A[3,3] & \cdots & A[3,q] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A[q,1] & A[q,2] & A[q,3] & \cdots & A[q,q] \end{pmatrix} \tag{3}$$

where $A[1,1]$, $A[1,2]$,..., $A[q,q]$ are $m \times m$ matrices. With these notations the partial transposition $t_1$ amounts to permuting matrices $A[\alpha, \beta]$ and $A[\beta, \alpha]$, for all these block matrices. We use the same notations as in refs. 18–20, that is, we introduce the two following transformations on matrix $R$, the matrix inverse $\hat{I}$ and the homogeneous matrix inverse $I$:

$$\hat{I}\colon R \to R^{-1}, \qquad I\colon R \to \det(R) \cdot R^{-1} \tag{4}$$

and we introduce the (generically) infinite order homogeneous, and inhomogeneous, transformations:

$$K = t_1 \cdot I, \qquad \hat{K} = t_1 \cdot \hat{I} \tag{5}$$

The *homogeneous* inverse $I$ is a *polynomial* transformation on each of the entries $m_{ij}$ of $R$, which associates to each $m_{ij}$ its corresponding cofactor. Transformation $\hat{I}$ is an *involution* ($\hat{I}^2 = \mathcal{I}$), whereas $I^2 = (\det(R))^{qm-2} \cdot \mathcal{I}$, where $\mathcal{I}$ denotes the identity transformation. Transformation $t_1$ is also an involution. The transformation $\hat{K}$ is clearly a *birational transformation* on the entries $m_{ij}$ since its inverse transformation is $\hat{I} \cdot t$ which is obviously also a rational transformation. Transformation $K$ is a *homogeneous polynomial transformation* on the entries $m_{ij}$ of the $R$-matrix.

For such vertex models of lattice statistical mechanics, transformations $\hat{I}$ and $t_1$ come from the *inversion relations*,[21, 22] and the geometrical symmetries of the lattice, in the framework of integrability and *beyond* integrability. These involutions generate a *discrete* group of (*birational*) automorphisms of the Yang–Baxter equations[4, 5] and their higher dimensional generalizations.[17] They also generate a discrete group of (birational) automorphisms of the algebraic varieties canonically associated with the Yang–Baxter equations (or their higher dimensional generalizations).[2] In the generic case where the birational transformation $\hat{K} = t_1 \cdot \hat{I}$ is an infinite order transformation, this set of birational automorphisms corresponds, essentially, to the iteration of $\hat{K}$. When the infinite order transformation $\hat{K}$ densifies algebraic varieties,[16] one can deduce the equations of these algebraic varieties exactly and *thus solve* the so-called "Baxterisation problem."

**Terminology:** In the following, we will call "$R$-matrix" the matrix we iterate with (5), even if this "$R$-matrix" does not satisfy a Yang–Baxter relation. From now on, we will call "*Baxterisation*" the procedure which amounts, for a given $R$-matrix, to iterating transformation (5), and finding the algebraic variety which contains all the points of this iteration. A good example corresponds, for instance, to get, from isolated $R$-matrices satisfying the tetrahedron relations,[23] $R$-matrices also solutions of the tetrahedron relations *but now depending on spectral parameters.*

## 3. BAXTERISATION: A WIN–WIN STRATEGY

When one "Baxterises" an $R$-matrix one can get three different kinds of situations: either the orbits of (5) are *chaotic*, and therefore one *cannot* expect any "nice" parametrization of the lattice model, or the orbits correspond to algebraic varieties, and one can actually introduce some "well-suited" parametrization of the model associated to these algebraic varieties, which will be extremely precious for any further analysis of the model (analysis of the Yang–Baxter equations, or higher dimensional generalizations, calculations of partition function per site,...), or, finally, these orbits are finite orbits. When the iteration of (5) is *infinite* and yields algebraic varieties, one can show that the algebraic varieties are not of the so-called "general-type"[2] (using the algebraic geometry terminology). For instance, for algebraic surfaces, one can actually classify the various surfaces that are not of the "general type:" product of elliptic curves, Enriques surfaces, Kummer surfaces, Abelian surfaces,.... When the orbits are *finite* one is back to algebraic varieties of the "general-type," which is a very large set of different situations, *quite hard to classify*.[2] However, writing for some integer $N$, the (projective) condition:

$$K^N(R) = \zeta \cdot R \tag{6}$$

*actually gives the equations* of the algebraic varieties corresponding to this very condition (6). For instance, one can find the algebraic subvariety of the chiral Potts model on which the higher genus Au-Yang–Baxter–Perk solutions[24] live, as a finite order condition.[11]

**To sum up:** one has the following situations:

- Either the group is *infinite*, and one gets the "precious" parametrization of the model (even if it is a parametrization in terms of theta functions of $g$ variables,...).

• The group is *finite*, and one actually gets the equations of the algebraic varieties corresponding to this situation, writing condition (6). This finite order situation is a "gold mine" as far as integrable situations are concerned.[6, 11] Let us recall, for instance, the example of the tetrahedron solutions.[6, 13]

• The group is *infinite*, but the orbits, corresponding to the iteration of (5), are *chaotic*: one does not have algebraic varieties[7] and therefore one does not have any Yang–Baxter structure.[8] This situation corresponds to an *exponential growth* of the iteration calculations.[25–27]

In every case the Baxterisation procedure helps to avoid the points of the parameter space, corresponding to exponential growth of the calculations, where no Yang–Baxter relation can be expected, and enables to get analytically the algebraic varieties corresponding to *finite order* iterations (see Section 6.1 below), or enables to actually build the algebraic varieties from the iteration of an *infinite order* birational transformation.

The Baxterisation procedure amounts to studying the iteration of a (resp. several) birational (resp. polynomial) mapping which actually corresponds to *discrete symmetries* of the parameter space of the models. It thus provides a natural link between lattice statistical mechanics (field theory,...) and the theory of *discrete dynamical systems*. Furthermore, it also provides links with many other domains of mathematical physics, or mathematics. For instance, as far as effective algebraic geometry, or even arithmetic, is concerned, the Baxterisation provides many natural, and simple, examples of *Abelian varieties* with an *infinite set of rational points*.[16, 28]

Let us first recall some well-known Yang–Baxter integrable situations corresponding to *finite order* conditions.

### 3.1. Finite Order Conditions: Free-Fermion Conditions

The matrix $R$ of the asymmetric eight-vertex model is of the form:

$$R = \begin{pmatrix} a & 0 & 0 & d' \\ 0 & b & c' & 0 \\ 0 & c & b' & 0 \\ d & 0 & 0 & a' \end{pmatrix} \tag{7}$$

[7] One may have algebraic *subvarieties* compatible with this situation, the subvarieties corresponding to the two previous situations.

[8] One can, however, try to characterize the "complexity" of this chaotic situation, calculating the *topological entropy*, or the Arnold complexity, of the birational mapping (5) one iterates.[25–27]

The free-fermion condition is:

$$aa' - dd' + bb' - cc' = 0 \tag{8}$$

A matrix of the form (7) may be brought, by similarity transformations, to a block-diagonal form:

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \qquad \text{with} \quad R_1 = \begin{pmatrix} a & d' \\ d & a' \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} b & c' \\ c & b' \end{pmatrix}$$

If one denotes by $\delta_1 = aa' - dd'$, and by $\delta_2 = bb' - cc'$, the determinants of the two blocks, then the homogeneous inverse $I$ (polynomial transformation) just reads:

$$\begin{array}{llll} a \rightarrow a' \cdot \delta_2, & a' \rightarrow a \cdot \delta_2, & d \rightarrow -d \cdot \delta_2, & d' \rightarrow -d' \cdot \delta_2 \\ b \rightarrow b' \cdot \delta_1, & b' \rightarrow b \cdot \delta_1, & c \rightarrow -c \cdot \delta_1, & c' \rightarrow -c' \cdot \delta_1 \end{array} \tag{9}$$

and transformation $t_1$ is given by: $t_1$: $c \leftrightarrow d'$ *and*: $d \leftrightarrow c'$. The condition (8) is *left invariant* by $t_1$, $I$, and thus $K = t_1 \cdot I$. It is straightforward to see that condition (8) is $\delta_1 = -\delta_2$ and has the effect of *linearizing* $I$ into:

$$\begin{array}{llll} a \rightarrow a', & a' \rightarrow a, & d \rightarrow -d, & d' \rightarrow -d' \\ b \rightarrow -b', & b' \rightarrow -b, & c \rightarrow c, & c' \rightarrow c' \end{array} \tag{10}$$

The group, generated by $I$ and $t_1$, is then realized by permutations of the entries, mixed with changes of signs, and its orbits are thus *finite*. The situation depicted here, namely an *inversion relation* that reduces, on some algebraic subvariety, to *permutation of the entries up to signs*, also occurs for *free-fermion* two-dimensional vertex models on a *triangular* lattice (see Sacco and Wu in ref. 29), or for the three-dimensional vertex corresponding to the Zamolodchikov–Baxter solution of the tetrahedron equations.[30–32] Details can be found in ref. 6.

The finite order situations are extremely favorable for Yang–Baxter integrability (chiral Potts model,[11] RSOS models,[7] tetrahedron solution,[6] free-fermion solutions, free para-fermions,...). If one tries to find new Yang–Baxter integrable models, one should certainly first try to write, systematically, all the algebraic subvarieties corresponding to these (projective) finite order conditions. However, in this finite order case, one may say that the "Baxterisation procedure" does not work, or "works to well:" it is too degenerate. As a consequence, many remarkable results, and structures, occur (polynomial representation of the natural integers together with their multiplication,...). These results, and structures, will be sketched in Section 6.1).

## 3.2. The Group Is Infinite: Iterations Associated with the Sixteen-Vertex Model

Let us now consider, with the example of the *sixteen vertex model*,[(33)] the situation[9] where $K$, or $\hat{K}$, is *infinite order*, thus yielding a *non-trivial* Baxterisation.

In the case of $4 \times 4$ matrices (see Fig. 1 but with $q = m = 2$), a particular permutation of the entries of the matrix, $t_1$, has been introduced in the framework of the symmetries of the *sixteen-vertex model*.[(34)] The action of two partial transpositions $t_1$, and $t_2$, on the $R$-matrix is given by:[(34)]

$$(t_1 R)^{ij}_{kl} = R^{kj}_{il}, \qquad (t_2 R)^{ij}_{kl} = R^{il}_{kj}, \qquad t = t_1 \cdot t_2 \tag{11}$$

Transformation $t$ is nothing but the matrix transposition: $t$ commutes with the matrix inversion $\hat{I}$. If one denotes $m_{ij}$ the entries of the $R$-matrix, this permutation corresponds to:

$$t_1 \colon m_{13} \leftrightarrow m_{31}, \quad m_{14} \leftrightarrow m_{32}, \quad m_{23} \leftrightarrow m_{41}, \quad m_{24} \leftrightarrow m_{42} \tag{12}$$

which amounts to permuting the two $2 \times 2$ (off-diagonal) sub-matrices of the $4 \times 4$ $R$-matrix. This transposition $t_1$ corresponds to a partial transposition of one direction (say the horizontal one denoted by "1," the other transposition $t_2$ corresponding to the other direction denoted by "2") of a two-dimensional vertex model[(4, 5, 34)] (see Fig. 1).

Remarkably, the symmetry group, generated by the matrix inverse $\hat{I}$ and transformation $t_1$, or by the infinite generator $\hat{K} = t_1 \cdot \hat{I}$, has been shown to yield *elliptic curves*[(5, 34)] which *foliate the whole parameter space* of the sixteen vertex model. One should not confuse the integrability of the *symmetries* of the parameter space of the sixteen vertex model (namely the mappings considered here) and the Yang–Baxter integrability:[(34)] the *sixteen vertex model is not generically Yang–Baxter integrable*.[10]

The integrability of the birational mapping $\hat{K}$, or, equivalently, of the homogeneous (bi-)polynomial transformation $K$, is closely related to the occurrence of *remarkable factorization schemes*.[(15, 16)] In order to see this, let us consider a $4 \times 4$ matrix $M_0 = R$, and the successive matrices obtained by iteration of transformation $K = t_1 \cdot I$, where $t_1$ is defined by (12).

[9] Generic from the point of view of the discrete dynamical systems.

[10] Using this elliptic parameterization,[(34)] one could imagine that the "inversion trick,"[(21, 22)] together with some "well-suited" analytical assumptions, could allow to actually get the exact expression of the partition function per site of the sixteen vertex model beyond the Yang–Baxter integrability. The model would be "calculable" *without being Yang–Baxter integrable*: this remains an open question.

Similarly to the factorizations described in refs. 15 and 16, one has, for arbitrary $n$, the following factorizations for the iterations of $K$:

$$\begin{aligned} M_{n+2} &= \frac{K(M_{n+1})}{f_n^2}, \qquad f_{n+2} = \frac{\det(M_{n+1})}{f_n^3} \\ \hat{K}_{t_1}(M_{n+2}) &= \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_{n+1} f_{n+3}} \end{aligned} \tag{13}$$

where the $f_n$'s are homogeneous polynomials in the entries of the initial matrix $M_0$ and the $M_n$'s are "reduced matrices" with homogeneous polynomial entries.[15, 16]

Let us denote by $\alpha_n$ the degree of the determinant of matrix $M_n$, and by $\beta_n$ the degree of polynomial $f_n$, and let us introduce $\alpha(x)$, $\beta(x)$ which are the generating functions of these $\alpha_n$'s, $\beta_n$'s:

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \qquad \beta(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n \tag{14}$$

From these factorizations one sees that one has a polynomial growth of the iteration calculations (quadratic growth of the degrees). Actually, one can easily get linear relations on the exponents $\alpha_n$, $\beta_n$ and exact expressions for their generating functions and for the $\alpha_n$'s and $\beta_n$'s:

$$\begin{aligned} \alpha(x) &= \frac{4(1+3x^2)}{(1-x)^3}, \qquad \beta(x) = \frac{4x}{(1-x)^3} \\ \alpha_n &= 4(2n^2+1), \qquad \beta_n = 2n(n+1) \end{aligned} \tag{15}$$

One has a whole hierarchy of recursions *integrable*, or compatible with integrability.[15] For instance, one has:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n+1} f_{n+4}^2 - f_{n+5} f_{n+2}^2}{f_n f_{n+4} f_{n+5} - f_{n+1} f_{n+2} f_{n+6}} \tag{16}$$

In an equivalent way, introducing the variable $x_n = \det(\hat{K}^n(R)) \cdot \det(\hat{K}^{n+1}(R))$, one gets a *hierarchy* of recursions on the $x_n$'s, (see ref. 15), the simplest recursion reading:

$$\frac{x_{n+2}-1}{x_{n+1}x_{n+2}x_{n+3}-1} = \frac{x_{n+1}-1}{x_n x_{n+1} x_{n+2}-1} \cdot x_n x_{n+1} x_{n+2}^2 \tag{17}$$

Equation (17) is equivalent to (16) since $x_n = (f_n^3 \cdot f_{n+2})/(f_{n+1}^3 \cdot f_{n-1})$. It can be seen that these recursions (16) and (17) are *integrable* ones.[15] For this, one can introduce[18] a new (homogeneous) variable:

$$q_n = \frac{f_{n+1} \cdot f_{n-1}}{f_n^2} \qquad \text{then} \quad x_n = \frac{q_{n+1}}{q_n} \tag{18}$$

and end up, after some simplifications, with the following biquadratic relation between $q_n$ and $q_{n+1}$:

$$q_n^2 \cdot q_{n+1}^2 + \mu \cdot q_n \cdot q_{n+1} + \rho \cdot (q_n + q_{n+1}) - \lambda = 0 \tag{19}$$

which is clearly an integrable recursion.[15] In terms of the $f_n$'s, the three parameters $\rho$, $\lambda$ and $\mu$, read:

$$\rho = \frac{f_1^{\,2} f_4 - f_2^{\,2} f_3}{f_1^3 f_3 - f_2^{\,3}}, \qquad \lambda = \frac{f_2 f_4 - f_1 f_3^{\,2}}{f_1^3 f_3 - f_2^3}, \qquad \mu = \frac{f_2^{\,5} - f_3^{\,2} f_1^{\,3} - f_1^{\,5} f_4 + f_3 f_2^{\,3}}{f_2 f_1 \cdot (f_1^3 f_3 - f_2^3)} \tag{20}$$

It may also be interesting to introduce:

$$\kappa = \frac{4 \cdot \lambda + \mu^2}{\rho} \tag{21}$$

For the most general sixteen vertex models the expressions of $\rho$, $\lambda$ and $\mu$ are quite large in terms of its sixteen homogeneous parameters and, thus, will not be given here. Let us just give an idea of these expressions in the simple Baxter limit:

$$\begin{aligned}
\rho &= (ba + cd)^2 \cdot (ba - cd)^2 \cdot (a^2 + b^2 - c^2 - d^2)^2, \qquad \lambda = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \\
\lambda_1 &= -(b^2 dc + b^2 a^2 + cda^2 - d^3 c + c^2 d^2 - c^3 d), \\
\lambda_2 &= (b^2 dc + b^2 a^2 + cda^2 - d^3 c - c^2 d^2 - c^3 d) \\
\lambda_3 &= (b^3 a + b^2 a^2 + ba^3 - d^2 ba - c^2 ba - c^2 d^2) \\
\lambda_4 &= (b^3 a - b^2 a^2 + ba^3 - d^2 ba - c^2 ba + c^2 d^2) \\
\mu &= -b^2 a^6 + 2a^4 b^2 c^2 - a^4 c^2 d^2 - 4a^4 b^4 + 2a^4 d^2 b^2 \\
&\quad - a^2 c^4 b^2 + 2a^2 c^4 d^2 + 2a^2 c^2 b^4 + 2a^2 c^2 d^4 \\
&\quad + 2d^4 b^2 c^2 - a^2 b^6 + 2a^2 b^4 d^2 - a^2 d^4 b^2 - c^6 d^2 \\
&\quad + 2c^4 d^2 b^2 - 4c^4 d^4 - d^6 c^2 - c^2 d^2 b^4
\end{aligned} \tag{22}$$

The analysis of these factorization schemes thus provides a simple complementary way of describing the Baxterisation and of finding the parametrization of the model (here elliptic curves). The equivalence between these elliptic curves (16), (17), (19), associated with the factorization scheme, and other elliptic curves, more closely related to the elliptic curves foliating the parameter space of the model, will be detailed in a forthcoming section (see Section 5.3).

## 4. BAXTERISATION OF MONODROMY MATRICES: $2M \times 2M$ MATRICES

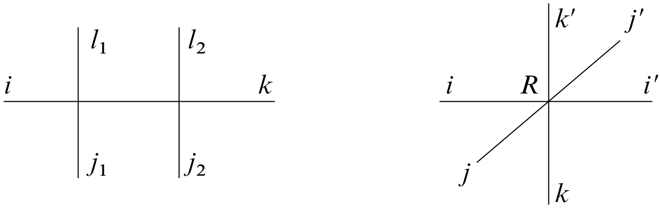
Let us now recall the more general vertex model (see Fig. 1 above), where the singled out (horizontal) direction (1) corresponds to a *two-dimensional* "auxiliary space" (that is $q=2$). The action of $t_1$, the "partial" transposition on the horizontal direction (1), is given by:[34]

$$(t_1 R)^{iJ}_{kL} = R^{kJ}_{iL} \qquad \text{that is} \quad t_1 \colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{23}$$

where $A$, $B$, $C$ and $D$ are $m \times m$ matrices. It is a straight calculation to see that the matrix inversion reads:

$$\hat{I} \colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} (A - B \cdot D^{-1} \cdot C)^{-1} & (C - D \cdot B^{-1} \cdot A)^{-1} \\ (B - A \cdot C^{-1} \cdot D)^{-1} & (D - C \cdot A^{-1} \cdot B)^{-1} \end{pmatrix} \tag{24}$$

This general framework enables *to take into account the analysis of* $N$-site monodromy[11] matrices[15] (take $m = 2^N$) of two-dimensional models, as well as the analysis of $d$-dimensional $2^d$-state *vertex models* (take $m = 2^{d-1}$). Let us just give here a pictorial representation of the two sites ($N=2$) monodromy matrix of a two-dimensional model and of a three-dimensional vertex model:



(25)

[11] This terminology of monodromy matrices in the framework of integrable lattice statistical mechanics, or quantum field theory, was introduced in the Quantum inverse Scattering Theory developed by the St Petersburg's School (see for instance, ref. 35). This matrix with operator entries is built as the product of local transition matrices along an auxiliary space, and thus generalizes the monodromy matrices occurring in the theory of classical inverse scattering.

Denoting $s = 2m$ the size of the matrices, the analysis of the corresponding factorizations yields for arbitrary $n$, "string-like" factorizations.[15, 16] For arbitrary $m$ (equal to $2^{d-1}$ or not), the analysis of the factorizations of the iterations of transformation $K$ yields:

$$M_1 = K(M_0), \qquad f_1 = \det(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1^{s-4}}, \qquad M_2 = \frac{K(M_1)}{f_1^{s-5}}$$

$$f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^{s-4}}, \qquad M_3 = \frac{K(M_2)}{f_1^5 \cdot f_2^{s-5}}$$

$$f_4 = \frac{\det(M_3)}{f_1^{2(s-4)} \cdot f_2^7 \cdot f_3^{s-4}}, \qquad M_4 = \frac{K(M_3)}{f_1^{2(s-5)} \cdot f_2^5 \cdot f_3^{s-5}}$$

$$f_5 = \frac{\det(M_4)}{f_1^8 \cdot f_2^{2(s-4)} \cdot f_3^7 \cdot f_4^{s-4}} \cdots$$

and, for arbitrary $n$, the following 'string-like" factorizations:

$$\begin{aligned} K(M_n) &= M_{n+1} \cdot f_n^{s-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{2(s-5)} \cdot f_{n-3}^6 \cdot f_{n-4}^{2(s-5)} \cdot f_{n-5}^6 \cdots \\ \det(M_n) &= f_{n+1} \cdot f_n^{s-4} \cdot f_{n-1}^7 \cdot f_{n-2}^{2(-s-4)} \cdot f_{n-3}^8 \cdot f_{n-4}^{2(s-4)} \cdot f_{n-5}^8 \cdot f_{n-6}^{2(s-4)} \cdots \end{aligned} \tag{26}$$

One easily gets from (26):

$$\begin{aligned} \alpha(x) &= \frac{s}{1+x} + s^2 \frac{x(1+x^2)}{(1+x)(1-x)^4}, \qquad \beta(x) = \frac{sx}{(1-x)^3} \\ \alpha_n &= \frac{s}{3}(2n+1)(2n^2+2n+3), \qquad \beta_n = \frac{s}{2} n(n+1) \end{aligned} \tag{27}$$

The $\alpha_n$'s and $\beta_n$'s are, respectively, *cubic and quadratic functions* of $n$.

### 4.1. Towards Bethe Ansatz: The Propagation Property

This *polynomial growth* of the calculation can be understood as follows. One of the "keys" to the Bethe Ansatz is the existence (see equations (B.10), (B.11a) in ref. 36) of vectors which are *pure tensor products* (of the form $v \otimes w$) and which $R$ maps onto pure tensor product $v' \otimes w'$. This key property[12] was called *propagation property* by R. J. Baxter, and

[12] Which is "almost" a sufficient condition for the Yang–Baxter equation. In the case of the Baxter model this non trivial relation corresponds to some intertwining relation of the product of two theta functions, which is nothing but the *quadratic Frobenius relations* on theta functions.[37, 38]

corresponds to the existence of a *Zamolodchikov algebra*[39] for the Baxter model.[13] This "propagation" equation reads here:

$$R(u \otimes V) = \mu \cdot u' \otimes V' \qquad \text{with} \quad u = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad u' = \begin{pmatrix} 1 \\ p' \end{pmatrix} \tag{28}$$

vectors $V$ and $V'$ having $m$ coordinates. One can rewrite (28) under the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V \\ p \cdot V \end{pmatrix} = \mu \cdot \begin{pmatrix} V' \\ p' \cdot V' \end{pmatrix} \tag{29}$$

Actually, for all the vertex models for which transposition $t_1$ can be represented as (23) (namely monodromy matrices, or $d$-dimensional vertex models, with "arrows" taking *two* colors,...), one can associate an *algebraic curve* of equation:

$$\det(Ap' - C - Dp + pp'B) = 0 \tag{30}$$

which form is invariant by $t_1$, $\hat{I}$ and thus by $\hat{K}$ or $\hat{K}^2$. As a byproduct this provides a *canonical Jacobian variety for such vertex models*, namely the Jacobian variety associated with curve (30). This procedure, which associates with an $R$-matrix the algebraic curve (30), originates from a key "propagation" relation (28), closely related to the action of the birational transformations $\hat{K}$.[34]

Thus one sees that, even when the Baxterisation procedure does not yield (elliptic or rational) curves but yields higher dimensional Abelian varieties, the occurrence of polynomial growth of the iteration calculations can thus be seen as a simple "detector" of such quite involved parameterizations.

[13] The existence of a Zamolodchikov algebra is, at first sight, a *sufficient* condition for the Yang–Baxter equations to be verified. Theta functions of $g$ variables *do satisfy quadratic Frobenius relations*,[38] consequently *yielding a Zamolodchikov algebra parameterized in term of theta functions of several variables*. However this Zamolodchikov algebra is apparently *not sufficient* for Yang–Baxter equations to be satisfied.[40] For the corresponding vertex models ($g$ replicas of the Baxter model coupled together) it may be possible that the associated $R$-matrices, which are parameterized in term of theta functions of several variables (and thus correspond to a "nice" Baxterisation), do not satisfy the Yang–Baxter equations, but could be such that the partition function per site could be calculated exactly using the "inversion trick:" this remains an open question.

## 4.2. Continuous Symmetries Generalizing the Gauge Symmetries

The birational transformation $\hat{K} = t_1 \cdot \hat{I}$ can be represented as follows:

$$\hat{K} = t_1 \cdot \hat{I}: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$
$$= \begin{pmatrix} (A - B \cdot D^{-1} \cdot C)^{-1} & (B - A \cdot C^{-1} \cdot D)^{-1} \\ (C - D \cdot B^{-1} \cdot A)^{-1} & D - C \cdot A^{-1} \cdot B)^{-1} \end{pmatrix} \qquad (31)$$

Let us introduce the following $SL(m) \times SL(m)$ transformation $\mathcal{G}^{(m)}$:

$$\mathcal{G}^{(m)}: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} G_L^{(m)} \cdot A \cdot G_R^{(m)} & G_L^{(m)} \cdot B \cdot G_R^{(m)} \\ G_L^{(m)} \cdot C \cdot G_R^{(m)} & G_L^{(m)} \cdot D \cdot G_R^{(m)} \end{pmatrix}$$

where $G_L^{(m)}$ and $G_R^{(m)}$ are two $SL(m)$ matrices. It is straightforward to see that:

$$\hat{K} \begin{pmatrix} G_L^{(m)} \cdot A \cdot G_R^{(m)} & G_L^{(m)} \cdot B \cdot G_R^{(m)} \\ G_L^{(m)} \cdot C \cdot G_R^{(m)} & G_L^{(m)} \cdot D \cdot G_R^{(m)} \end{pmatrix}$$
$$= \begin{pmatrix} (G_R^{(m)})^{-1} \cdot A' \cdot (G_L^{(m)})^{-1} & (G_R^{(m)})^{-1} \cdot B' \cdot (G_L^{(m)})^{-1} \\ (G_R^{(m)})^{-1} \cdot C' \cdot (G_L^{(m)})^{-1} & (G_R^{(m)})^{-1} \cdot D' \cdot (G_L^{(m)})^{-1} \end{pmatrix} \qquad (32)$$

and, in a second step, that:

$$\mathcal{G}^{(m)}(\hat{K}^2(R)) = \hat{K}^2(\mathcal{G}^{(m)}(R)) \qquad (33)$$

In fact, such a result is *not specific of a two-dimensional auxiliary space.* Recalling the $(qm) \times (qm)$ $R$-matrix (3), and the associated partial transposition $t_1$, and introducing a $SL(m) \times SL(m)$ transformation $\mathcal{G}^{(m)}$ which transforms each $m \times m$ block $A[\alpha, \beta]$ into $G_L \cdot A[\alpha, \beta] \cdot G_R$, one recovers, again, relation (33). Of course there is nothing specific with $t_1$, and one finds the same results for the partial transposition $t_2$ corresponding to the vertical line, with this time, a commutation between $\hat{K}^2$ and a $SL(q) \times SL(q)$ transformation $\mathcal{G}^{(q)}$. Since $t_2 = t_1 \cdot t$, and since $t$ commutes with $\hat{I}$, one immediately gets[14] with obvious notations the following $SL(q) \times SL(m) \times SL(q) \times SL(m)$ symmetry for $\hat{K}^2$:

$$\hat{K}^2(G_L^{(q)} \otimes G_L^{(m)} \cdot R \cdot G_R^{(q)} \otimes G_R^{(m)}) = G_L^{(q)} \otimes G_L^{(m)} \cdot (\hat{K}^2(R)) \cdot G_R^{(q)} \otimes G_R^{(m)} \qquad (34)$$

[14] Similarly for a three-dimensional $2^3 \times 2^3$ $R$-matrix one gets easily $\hat{K}^2(g_{1L}^{-1} \otimes g_{2L}^{-1} \otimes g_{3L}^{-1} \cdot R_{3D} \cdot g_{1R} \otimes g_{2R} \otimes g_{3R}) = g_{1L}^{-1} \otimes g_{2L}^{-1} \otimes g_{3L}^{-1} \cdot \hat{K}^2(R_{3D}) \cdot g_{1R} \otimes g_{2R} \otimes g_{3R}$.

This symmetry drastically generalizes the well-known *gauge symmetries* introduced by Wegner[41] which correspond to similarity transformations $G_R^{(q)} = (G_L^{(q)})^{-1}$ and $G_R^{(m)} = (G_L^{(m)})^{-1}$. Let us note that this kind of symmetries, generalizing the gauge transformations, have actually been used (inhomogeneous gauge transformations in a Yang–Baxter framework) by R. J. Baxter to map the Baxter model onto an *inhomogeneous* six vertex model, in order to build the Bethe Ansatz of the Baxter model.[36]

## 5. SIXTEEN VERTEX MODEL AND BAXTER MODEL: REVISITING THE ELLIPTIC CURVES

Considering the (non-generically Yang–Baxter integrable) sixteen vertex model, one finds that a *canonical* parameterization in terms of *elliptic curves* occurs in the sixteen homogeneous parameter space of the model.[34] This canonical parameterization is obtained[15] from the *Baxterisation* procedure.[34] In fact several elliptic curves (associated with different "spaces") occur: one corresponding to the factorization analysis of Section 3.2, another from the propagation property (28), and another one from the iteration of the birational transformation $\hat{K}^2$ in the sixteen homogeneous parameter space of the model.[34] Let us analyse the relations between these various elliptic curves and show that they *actually identify*.

Let us recall the results and notations concerning the sixteen vertex model.[34] Let us use the following notation for $R$:

$$R = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \tag{35}$$

### 5.1. Propagation Property for the Sixteen Vertex Model and the Baxter Model

Considering the sixteen vertex model (35), the propagation relation (28), for $m = 2$, becomes[16] $R(v_n \otimes w_n) = \mu \cdot v_{n+1} \otimes w_{n+1}$ where:

$$v_n = \begin{pmatrix} 1 \\ p_n \end{pmatrix}, \qquad w_n = \begin{pmatrix} 1 \\ \tilde{p}_n \end{pmatrix}, \qquad v_{n+1} = \begin{pmatrix} 1 \\ p_{n+1} \end{pmatrix}, \qquad w_{n+1} = \begin{pmatrix} 1 \\ \tilde{p}_{n+1} \end{pmatrix} \tag{36}$$

[15] In the Yang–Baxter integrable subcase, the Baxter model, this elliptic parameterization, deduced from the Baxterisation procedure, is actually the elliptic parameterization introduced by R. J. Baxter to solve the Baxter model.[42]

[16] See the propagation property for the Baxter model, namely (B.10), (B.11a) in ref. 36.

and yields, by eliminations of $p_n$, $p_{n+1}$ (resp. $\tilde{p}_n$, $\tilde{p}_{n+1}$), the two biquadratic relations:[34]

$$l_4 + l_{11} \cdot p_n - l_{12} \cdot p_{n+1} + l_2 \cdot p_n^2 + l_1 \cdot p_{n+1}^2 - (l_9 + l_{18}) \cdot p_n \cdot p_{n+1} - l_{13} \cdot p_n^2 \cdot p_{n+1} + l_{10} \cdot p_n \cdot p_{n+1}^2 + l_3 \cdot p_n^2 \cdot p_{n+1}^2 = 0 \tag{37}$$

$$l_7 + l_{16} \cdot \tilde{p}_n - l_{15} \cdot \tilde{p}_{n+1} + l_8 \cdot \tilde{p}_n^2 + l_5 \cdot \tilde{p}_{n+1}^2 - (l_9 - l_{18}) \cdot \tilde{p}_n \cdot \tilde{p}_{n+1} - l_{17} \cdot \tilde{p}_n^2 \cdot \tilde{p}_{n+1} + l_{14} \cdot \tilde{p}_n \tilde{p}_{n+1}^2 + l_6 \tilde{p}_n^2 \tilde{p}_{n+1}^2 = 0 \tag{38}$$

where the $l_i$'s are quadratic expressions of the entries (35) of the $R$-matrix.[34] These two biquadratics can be seen as:

$$[p'^2, p', 1] \cdot R_3^{(1)} \cdot \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0 \qquad \text{and} \qquad [q'^2, q', 1] \cdot R_3^{(2)} \cdot \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix} = 0$$

where the two $3 \times 3$ matrices read:

$$R_3^{(1)} = \begin{bmatrix} l_1 & l_{10} & l_3 \\ -l_{12} & -(l_9 + l_{18}) & -l_{13} \\ l_4 & l_{11} & l_2 \end{bmatrix} \qquad \text{and}$$

$$R_3^{(2)} = \begin{bmatrix} l_5 & l_{14} & l_6 \\ -l_{15} & -(l_9 - l_{18}) & -l_{17} \\ l_7 & l_{16} & l_8 \end{bmatrix} \tag{39}$$

In the Baxter limit these two matrices reduce to a *only one* $3 \times 3$ matrix:

$$R_3^{bax} = \begin{bmatrix} J_x + J_y & 0 & J_x - J_y \\ 0 & 4J_z & 0 \\ J_x - J_y & 0 & J_x + J_y \end{bmatrix} \tag{40}$$

where $J_x$, $J_y$, and $J_z$ are the three well-known quadratic expressions of the $XYZ$ Hamiltonian:

$$J_x = a \cdot b + c \cdot d, \qquad J_y = a \cdot b - c \cdot d, \qquad J_z = \frac{a^2 + b^2 - c^2 - d^2}{2} \tag{41}$$

Some $\hat{K}^2$-invariants can be deduced from $SL(3)$ invariants of the two $3 \times 3$ matrices (39), namely a quadratic expression in the $l_i$'s (for $R_3^{(1)}$ for instance) $l_1 \cdot l_2 + l_3 \cdot l_4 - l_{10} \cdot l_{11} - l_{12} \cdot l_{13} + (l_9 + l_{18})^2$, a cubic (in the $l_i$'s) which is nothing but their *determinant*, and a quartic one. Eighteen (algebraically related) *quadratic* polynomials ($p_1$,..., $p_{18}$) which are linear combinations of the $l_i$'s, and transform very simply under $t_1$ and $I$, have been found.(34) Introducing the ratio of these covariants $p_i$'s, one gets *invariants* of $\hat{K}^2$ thus giving the equations of the elliptic curves: the elliptic curves are given by the intersection of fourteen quadrics.(34)

### 5.2. Reduction of a Sixteen Vertex Model to a $K^2$-Effective Baxter Model

From (34) for $q = m = 2$, one deduces a $sl_2 \times sl_2 \times sl_2 \times sl_2$ symmetry on the sixteen vertex model.(34) Furthermore, the $R$-matrix of the sixteen vertex *can actually be decomposed*[17] as:

$$R_{\text{sixteen}} = g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot R_{\text{Baxter}} \cdot g_{1R} \otimes g_{2R} \tag{42}$$

where $R_{\text{Baxter}}$ denotes the $R$-matrix of an "effective" Baxter model and $g_{1R}$, $g_{2R}$, $g_{1L}$, $g_{2L}$ are $2 \times 2$ matrices. The sixteen homogeneous parameters of the sixteen vertex are thus decomposed into four homogeneous parameters of an "effective" Baxter model and four times three parameters (four homogeneous parameters) of the various $2 \times 2$ matrices: $g_{1R}$, $g_{2R}$, $g_{1L}$, $g_{2L}$. Using this very decomposition (42), and the previous symmetry relation (34) for $q = m = 2$, one actually gets:

$$\hat{K}^2(R_{\text{sixteen}}) = g_{1L}^{-1} \otimes g_{2L}^{-1} \cdot \hat{K}^2(R_{\text{Baxter}}) \cdot g_{1R} \otimes g_{2R} \tag{43}$$

The matrices $g_{1R}$, $g_{2R}$, $g_{1L}$, $g_{2L}$ of the decomposition (42) can thus be seen as *constants of motion* of the iteration of $\hat{K}^2$.

If $R_{\text{Baxter}}$ belongs to a "special" manifold, or algebraic variety, $R_{\text{sixteen}}$, given by (42), will *also belong* to a "special" manifold, or algebraic variety: for instance, if $R_{\text{Baxter}}$ belongs to a *finite order* algebraic variety for the iteration of $\hat{K}^2$, namely $\hat{K}^{2N}(R_{\text{Baxter}}) = \eta \cdot R_{\text{Baxter}}$, then $R_{\text{sixteen}}$ will also belong to a finite order algebraic variety for the iteration of $\hat{K}^2$: $\hat{K}^{2N}(R_{\text{sixteen}}) = \eta \cdot R_{\text{sixteen}}$. If $R_{\text{Baxter}}$ belongs to a *critical variety* then $R_{\text{sixteen}}$ given by (42) should also belong to a critical variety. This last result does not come from the fact that $g_{1L}$, $g_{2L}$, $g_{1R}$, $g_{2R}$ are symmetries of the partition function (they are not, except in the gauge case: $g_{1L} = g_{1R}$ with $g_{2L} = g_{2R}$): they are

[17] Finding, for a given $R$-matrix of the sixteen vertex model, the elements of this decomposition, namely $R_{\text{Baxter}}$ and $g_{1R}$, $g_{2R}$, $g_{1L}$, $g_{2L}$, is an extremely difficult process that will not be detailed here. Conversely, one can show easily that the matrices of the form (42) *span the whole space* of $4 \times 4$ matrices.

symmetries of $\hat{K}^2$ which is a *symmetry of the critical manifolds*. Therefore they are *symmetries of the critical manifolds even if they are not symmetries of the partition function.*

A decomposition, like (42), is closely associated to the *parametrization of the sixteen vertex model* in *elliptic curves*:[34] given $R_{\text{Baxter}}$, $g_{1L}$, $g_{2L}$, $g_{1R}$ and $g_{2R}$, one can easily deduce $R_{\text{sixteen}}$. Conversely, given $R_{\text{sixteen}}$, it is extremely difficult to get $R_{\text{Baxter}}$, $g_{1L}$, $g_{2L}$, $g_{1R}$ and $g_{2R}$, however, and remarkably, it is quite simple to get $R_{\text{Baxter}}$. Since $g_{1L}$, $g_{1R}$, $g_{2L}$, $g_{2R}$ are $\hat{K}^2$-invariants, one can try to relate, directly, the "$\hat{K}^2$-effective" covariants $J_x$, $J_y$ and $J_z$ with the $\hat{K}^2$-invariants related to the recursion on the $x_n$'s or the $q_n$'s, namely $\rho$, $\mu$, $\lambda$, or $\kappa$ (see (19), (21)). In terms of these well-suited algebraic covariants, the previous parameters read:

$$\rho = 4J_z^2 J_x^2 J_y^2, \qquad \mu = -2 \cdot (J_z^2 J_x^2 + J_z^2 J_y^2 + J_x^2 J_y^2), \qquad \kappa = 4 \cdot (J_z^2 + J_x^2 + J_y^2)$$
$$\lambda = -(J_z^2 J_x^2 + J_z^2 J_y^2 + J_x^2 J_y^2)^2 + 4 \cdot (J_z^2 + J_x^2 + J_y^2) \cdot J_z^2 J_x^2 J_y^2 \qquad (44)$$

One immediately recognizes some symmetric polynomials of $J_x^2$, $J_y^2$, $J_z^2$. Therefore it is easy to nee that the $J_x$, $J_y$, $J_z$ can be straightforwardly obtained from a cubic polynomial $P(u)$:

$$P(u) = 4 \cdot u^3 - \kappa \cdot u^2 - 2 \cdot \mu \cdot u - \rho = 4 \cdot (u - J_x^2) \cdot (u - J_y^2) \cdot (u - J_z^2) \qquad (45)$$

This is remarkable, because trying to get $J_x$, $J_y$, $J_z$, by brute-force eliminations from (42), yields huge calculations. In fact, one only gets, from (45), the *squares* of the $J_x$, $J_y$, $J_z$, but the critical manifold, as well as the finite order conditions (see below), only depend on $J_x^2$, $J_y^2$, $J_z^2$.

The elliptic curves corresponding to the orbits of $\hat{K}^2$ in the parameter space of the sixteen vertex model, as well as the two biquadratics (37) and (38), together with the elliptic curves associated to the factorization analysis of Section 3.2 (like (16), (17) or (19)), share the *same* modular invariant $j$, which can be simplify written, for the sixteen vertex model, in terms of the "effective" $J_x^2$, $J_y^2$, $J_z^2$ deduced from (45) (using of course (20)):

$$j = 256 \, \frac{(J_x^4 + J_y^4 + J_z^4 - J_z^2 J_y^2 - J_z^2 J_x^2 - J_y^2 {J^2}_x)^3}{(J_y^2 - J_x^2)^2 \, (J_z^2 - J_x^2)^2 \, (J_z^2 - J_y^2)^2} \qquad (46)$$

Based on the classical theory of algebraic invariants (see ref. 43) an irreducible basis of algebraic invariants have been built for the sixteen vertex model:[44–53] these algebraic invariants take into account the weak-graph "gauge" (similarity...) $sl_2 \times sl_2$ symmetries of the sixteen vertex model.[41] Beyond these similarity symmetries, *inhomogeneous* gauge transformations were even considered, thus introducing much larger set of $sl_2 \times sl_2 \times sl_2 \times sl_2$

symmetries.[44–53] The modular invariant $j$ is, of course, invariant under the previous $sl_2 \times sl_2$ similarity symmetries, but it is actually *also invariant*[34] under the much larger set of $sl_2 \times sl_2 \times sl_2 \times sl_2$ symmetries[18] described in Section 4.2. Furthermore this modular invariant is also invariant under the infinite discrete set of birational transformations $\hat{K}^N$, corresponding to the Baxterisation. It is thus invariant under a *continuous* group of *linear* transformations $sl_2 \times sl_2 \times sl_2 \times sl_2$ and, in the same time, under an *infinite discrete* group of *non-linear* transformations. In a forthcoming Section 6 it will also be seen to be invariant under another remarkable *infinite set of* (*non-invertible*) *polynomial transformations*. The modular invariant $j$ thus "encapsulates" *all* the symmetries of the sixteen vertex model. It can be calculated directly in terms of the $\mu$, $\rho$ and $\lambda$:

$$j = -\frac{(\mu^4 + 8\mu^2\lambda + 16\lambda^2 + 24\rho^2\mu)^3}{\rho^4(\mu^4\lambda + 8\mu^2\lambda^2 + 16\lambda^3 + \rho^2\mu^3 + 36\rho^2\mu\lambda + 27\rho^4)} \tag{47}$$

For the sixteen vertex model, using (20), the modular invariant $j$ becomes, in terms of the sixteen homogeneous parameters of the model, the ratio of two "huge" homogeneous polynomials.

### 5.3. Biquadratic (19) Versus Biquadratic (35)

The Baxterisation process is associated with the iteration of $\hat{K}$, or, rather, $\hat{K}^2$. Since, as far as $\hat{K}^2$ is concerned, one can reduce a sixteen vertex model to an "effective" Baxter model, one can try to revisit, directly, the relation between the biquadratic (19) and the "propagation curve" (37) for the Baxter model. For the Baxter model, relation (19) becomes the biquadratic:

$$\begin{aligned} q_n^2 q_{n+1}^2 &- 2 \cdot (J_z^2 J_x^2 + J_z^2 J_y^2 + J_x^2 J_y^2) \cdot q_n \cdot q_{n+1} + 4 J_z^2 J_x^2 J_y^2 \cdot (q_n + q_{n+1}) \\ &+ (J_z^2 J_x^2 + J_z^2 J_y^2 + J_x^2 J_y^2)^2 - 4 \cdot (J_z^2 + J_x^2 + J_y^2) \cdot J_z^2 J_x^2 J_y^2 = 0 \end{aligned} \tag{48}$$

which should be compared with the "propagation" biquadratics (37) of the Baxter model:[36, 42]

$$\begin{aligned} \Gamma_1(p_n, p_{n+1}) &= (J_x - J_y) \cdot (p_n^2 p^2{}_{n+1} + 1) - (J_x + J_y) \cdot (p_n^2 + p_{n+1}^2) \\ &\quad + 4 \cdot J_z \cdot p_n \cdot p_{n+1} \\ &= (p_n^2 - 1) \cdot (p_{n+1}^2 - 1) \cdot J_x - (p_n^2 + 1) \cdot (p_{n+1}^2 + 1) \cdot J_y \\ &\quad + 4 \cdot J_z \cdot p_n \cdot p_{n+1} = 0 \end{aligned} \tag{49}$$

[18] These symmetries are not invariance of the partition function, like the previous similarity symmetries,[41, 44] but only symmetries of the parameter space.

It is known that simple "propagation" curves, like (49), have the following elliptic parameterization:[36, 42]

$$p_n = sn(u_n, k), \qquad p_{n+1} = sn(u_{n+1}, k), \qquad \text{where} \quad u_{n+1} = u_n \pm \lambda \tag{50}$$

where $sn(u, k)$ denotes the elliptic sinus of modulus $k$ and $\lambda$ now denotes some "shift." The modulus[54] $k$ is equivalent to the following modulus which has a very simple expression in terms of $J_x^2$, $J_y^2$ and $J_z^2$:

$$\mathcal{M} = \frac{J_z^2 - J_y^2}{J_x^2 - J_z^2} \tag{51}$$

At first sight it seems that one has two different elliptic curves (biquadratics), namely (48) which is symmetric under permutations of $J_x^2$, $J_y^2$ and $J_z^2$, and (49) which *breaks this symmetry*. Let us also consider the same "propagation curve" (49), but now between $p_{n+1}$ and $p_{n+2}$, and let us eliminate $p_{n+1}$ between these two algebraic curves. One gets, after the factorization of $(p_n - p_{n+1})^2$:

$$\begin{aligned}
\Gamma_2(p_n, p_{n+2}) &= 2J_z^2 \cdot (J_y^2 - J_x^2) \cdot (p_n^2 p_{n+2}^2 + 1) + 2J_x^2 J_y^2 \cdot (p_n^2 + p_{n+2}^2) \\
&\quad + 4 \cdot (J_x^2 J_y^2 - J_z^2 J_x^2 - J_z^2 J_y^2) \cdot p_n \cdot p_{n+2} \\
&= (p_n^2 - 1) \cdot (p_{n+2}^2 - 1) \cdot J_x^{(2)} - (p_n^2 + 1) \cdot (p_{n+2}^2 + 1) \\
&\quad \cdot J_y^{(2)} + 4 \cdot J_z^{(2)} \cdot p_n \cdot p_{n+2} = 0
\end{aligned} \tag{52}$$

where $J_x^{(2)}$, $J_y^{(2)}$ and $J_z^{(2)}$ are given below (see (55)). One remarks that (52) is actually of the *same form* as (49). The two biquadratic curves (48) and (52) are (birationally) equivalent, their shift $\lambda$ and modular invariant[54] being equal. Actually one can find directly the homographic transformation

$$q_n = \frac{\alpha \cdot p_n + \beta}{\gamma \cdot p_n + \delta}, \qquad q_{n+1} = \frac{\alpha \cdot p_{n+2} + \beta}{\gamma \cdot p_{n+2} + \delta} \tag{53}$$

which maps (48) onto (52), the parameters $\alpha, \ldots \delta$ of the homographic transformation (53) being quite involved. The calculations are quite tedious and will be given elsewhere.

Let us make a few comments on this equivalence between the biquadratic (48), which is *symmetric under the permutations* of $J_x$, $J_y$ and $J_z$, and the biquadratic (52) which breaks this symmetry. In fact, the two elliptic curves (48) and (52) share the same modular invariant (46), symmetric under the permutations of $J_x$, $J_y$ and $J_z$. This modular invariant can be written:

$$j = 256\,\frac{(1-\mathcal{M}+\mathcal{M}^2)^3}{\mathcal{M}^2(1-\mathcal{M})^2} \tag{54}$$

where $\mathcal{M}$ is given by the simple expression (51), but *also* similar expressions where $J_x$, $J_y$ and $J_z$ are permuted (which amounts to performing simple homographic changes on $\mathcal{M}$ representing the permutation group of three elements, namely $\mathcal{M} \to 1/\mathcal{M}$, $\mathcal{M} \to 1-\mathcal{M}$,...). Similarly, the biquadratic (52) is (birationally) equivalent to five other equivalent biquadratics deduced from (52) by permutations of $J_x$, $J_y$ and $J_z$.

The previously described elimination of $p_{n+1}$, changing $\Gamma_1$ into $\Gamma_2$, amounts to eliminating $u_{n+1}$ between $u_n \to u_{n+1} = u_n \pm \lambda$ and $u_{n+1} \to u_{n+2} = u_{n+1} \pm \lambda$ thus getting $u_n \to u_{n+2} = u_n \pm 2\cdot\lambda$, together with two times $u_n \to u_{n+2} = u_n$. Considering the coefficients of the biquadratic (52) one thus gets, very simply, a *polynomial representation* of the shift doubling $\lambda \to 2\cdot\lambda$:

$$\begin{aligned}
J_x \to J_x^{(2)} &= -J_z^2J_x^2 + J_z^2J_y^2 - J_x^2J_y^2\\
J_y \to J_y^{(2)} &= J_z^2J_x^2 - J_z^2J_y^2 - J_x^2J_y^2\\
J_z \to J_z^{(2)} &= -J_z^2J_x^2 - J_z^2J_y^2 + J_x^2J_y^2
\end{aligned} \tag{55}$$

The modulus (51) is (as it should) *invariant* by (55), which represents the shift doubling transformation. A general calculation, corresponding to eliminations between two biquadratics $\Gamma_1$ of same modulus (51), but different shifts $\lambda$ and $\lambda'$, will be given elsewhere.

Of course there is nothing specific with the shift doubling: similar calculations can be performed to get *polynomial representations* of $\lambda \to M\cdot\lambda$, for *any integer* $M$. Actually, it will be seen on many examples given in the next section and in Appendix A, that the multiplication of the shift by a *prime* number $N \neq 2$ has the following *polynomial representation* $(J_x, J_y, J_z) \to (J_x^{(N)}, J_y^{(N)}, J_z^{(N)})$:

$$\begin{aligned}
J_x^{(N)} &= J_x \cdot P_x^{(N)}(J_x, J_y, J_z)\\
J_y^{(N)} &= J_y \cdot P_y^{(N)}(J_x, J_y, J_z) = J_y \cdot P_x^{(N)}(J_y, J_z, J_x)\\
J_z^{(N)} &= J_z \cdot P_z^{(N)}(J_x, J_y, J_z) = J_z \cdot P_x^{(N)}(J_z, J_x, J_y)
\end{aligned} \tag{56}$$

where the $P_x^{(N)}(J_x, J_y, J_z)$'s (and thus $P_y^{(N)}(J_x, J_y, J_z)$, $P_z^{(N)}(J_x, J_y, J_z)$) are polynomials of $J_x^2$, $J_y^2$ and $J_z^2$.

**Terminology problem:** From the point of view of discrete dynamical systems a mapping like (55) (or (56) and the mappings given below (58),...) could, at first sight, be called "integrable:" the iteration of this (two-dimensional) mapping "densifies" algebraic curves (conics) foliating the whole two-dimensional space, namely $\mathcal{M} = constant$, exactly as an integrable mapping does.[18, 20] One can even write *explicit analytical expressions* for the $N$th iterate, *for any N*. However, this mapping is *not reversible*, the growth of the calculations[25–27] is exponential ($2^N$ *exponential growth*, ln(2) topological entropy,...). In fact this very example of "calculable" chaos is the exact equivalent of the situation encountered with the logistic map $x \to \alpha \cdot x \cdot (1 - x)$ for $\alpha = 4$: one does not have a representation of a translation $\theta \to \theta + N \cdot \lambda$, but a representation of the iteration of a multiplication by 2: $\theta \to 2^N \cdot \theta$.

## 6. POLYNOMIAL REPRESENTATIONS OF THE MULTIPLICATION OF THE SHIFT BY AN INTEGER AND ASSOCIATED FINITE ORDER CONDITIONS

The multiplication of the shift by three can be obtained using the previous elimination procedure, namely eliminating $y$ between $\Gamma_2(x, y)$ and $\Gamma_1(y, z)$ (or equivalently eliminating $y$ between $\Gamma_1(x, y)$ and $\Gamma_2(y, z)$), thus yielding a resultant which factorizes into two biquadratics of the same form as the two previous ones, namely $\Gamma_1(x, z)$ and $\Gamma_3(x, z)$:

$$\Gamma_3(x, z) = (x^2 - 1) \cdot (z^2 - 1) \cdot J_x^{(3)} - (x^2 + 1) \cdot (z^2 + 1) \cdot J_y^{(3)} + 4 \cdot J_z^{(3)} \cdot x \cdot z = 0 \tag{57}$$

where $J_x^{(3)}$, $J_y^{(3)}$ and $J_z^{(3)}$ are polynomials in $J_x$, $J_y$ and $J_z$. This provides a *polynomial representation* $(J_x, J_y, J_z) \to (J_x^{(3)}, J_y^{(3)}, J_z^{(3)})$ of the multiplication of the shift by three. This polynomial representation is of the form (56), where

$$P_x^{(3)} = -2J_z^2 J_y^2 J_x^4 - 3J_y^4 J_z^4 + 2J_y^2 J_z^4 J_x^2 + J^4{}_y J_x^4 + 2J_y^4 J_z^2 J_x^2 + J_z^4 J_x^4 \tag{58}$$

The modulus (51) is (as it should) invariant by the polynomial representation (58) of the multiplication of the shift by three (58).

The multiplication of the shift by four has the following polynomial representation $(J_x, J_y, J_z) \to (J_x^{(4)}, J_y^{(4)}, J_z^{(4)})$:

$$\begin{aligned}
J_x^{(4)} = & -4J_x^6J_y^6J_z^4 - 6J_z^8J_y^4J_x^4 + 4J_z^8J_y^6J_x^2 \\
& + 4J_z^8J_y^2J_x^6 + 4J_z^6J_y^6J_x^4 - 4J_z^2J_x^8J_y^6 + 10J_z^4J_x^8J_y^4 \\
& - 4J_z^6J_x^8J_y^2 - J_x^8J_y^8 - J_z^8J_y^8 - J_z^8J_x^8 \\
& + 4J_z^2J^8{}_yJ_x^6 - 4J_z^6J_x^6J_y^4 + 4J_z^6J_y^8J_x^2 - 6J_z^4J_y^8J_x^4 \\
J_y^{(4)} = & - J_x^8J_y^8 - J_z^8J_y^8 - J_z^8J_x^8 - 4J_x^6J_y^6J_z^4 \\
& - 6J_z^8J_y^4J_x^4 + 4J_z^8J_y^6J_x^2 + 4J_z^8J_y^2J_x^6 - 4J_z^2J_y^8J_x^6 \\
& + 10J_z^4J_y^8J_x^4 - 4J_z^6J_y^6J_x^4 - 4J_z^6J_y^8J_x^2 \\
& + 4J_z^2J_x^8J_y^6 - 6J_z^4J_x^8J_y^4 + 4J_z^6J_x^8J_y^2 + 4J_z^6J_x^6J_y^4 \\
J_z^{(4)} = & - J_x^8J_y^8 - J_z^8J_y^8 - J_z^8J_x^8 - 6J_z^4J_x^8J_y^4 \\
& + 4J_z^6J_x^8J_y^2 + 4J_z^2J_y^8J_x^6 - 4J_z^6J_y^6J_x^4 + 10J_z^8J^4{}_yJ_x^4 \\
& - 4J_z^8J_y^6J_x^2 - 4J_z^8J_y^2J_x^6 + 4J_x^6J_y^6J_z^4 \\
& - 4J_z^6J_x^6J_y^4 + 4J_z^2J_x^8J_y^6 + 4J_z^6J_y^8J_x^2 - 6J_z^4J_y^8J_x^4
\end{aligned} \tag{59}$$

which can be obtained, either by the elimination of $y$ between $\Gamma_2(x, y)$ and $\Gamma_2(y, z)$ (and extracting a $(x-z)^2$ factor in the resultant), or, equivalently, by the elimination of $y$ between $\Gamma_1(x, y)$ and $\Gamma_3(y, z)$, or the elimination of $y$ between $\Gamma_3(x, y)$ and $\Gamma_1(y, z)$ (and extracting a $\Gamma_2$ factor in the resultant). Again, one gets $\Gamma_4(x, z)$:

$$\Gamma_4(x, z) = (x^2-1)\cdot(z^2-1)\cdot J_x^{(4)} - (x^2+1)\cdot(z^2+1)\cdot J_y^{(4)} + 4\cdot J_z^{(4)}\cdot x\cdot z = 0 \tag{60}$$

where $J_x^{(4)}$, $J_y^{(4)}$ and $J_z^{(4)}$ are given above. It can easily be verified that (59) can be obtained directly combining (55) with itself.

The multiplication of the shift by five has a polynomial representation $(J_x, J_y, J_z) \to (J_x^{(5)}, J_y^{(5)}, J_z^{(5)})$ of the form (56), where:

$$\begin{aligned}
& P_5(J_x, J_y, J_z) \\
& \quad = 5J_z{}^{12}J_y{}^{12} + (J_z{}^2 - J_y{}^2)^6\, J_x{}^{12} - 10J_y{}^{10}J_z{}^{10}(J_z{}^2 + J_y{}^2)\, J_x{}^2 \\
& \qquad + 36J_y{}^6J_z{}^6(J_z{}^2 + J_y{}^2)(J_z{}^2 - J_y{}^2)^2\, J_x{}^6 \\
& \qquad + J_y{}^8J_z{}^8(4J_zJ_y + 3J_z{}^2 - 3J_y{}^2)(3J_y{}^2 + 4J_zJ_y - 3J_z{}^2)\, J_x{}^4 \\
& \qquad - J_z{}^4J_y{}^4(29J_z{}^4 + 54J_z{}^2J_y{}^2 + 29J_y{}^4)(J_z{}^2 - J_y{}^2)^2\, J_x{}^8 \\
& \qquad + 2J_z{}^2J_y{}^2(J_y{}^2 + 3J_z{}^2)(3J_y{}^2 + J_z{}^2)(J_z{}^2 + J_y{}^2)(J_z{}^2 - J_y{}^2)^2\, J_x{}^{10}
\end{aligned} \tag{61}$$

The modulus (51) is, again, invariant by this last polynomial representation of the multiplication of the shift by five. One remarks that $P_x^{(5)}(J_x, J_y, J_z)$ singles out $J_x$ and is invariant under the permutation $J_y \leftrightarrow J_z$ and, similarly, $P_y^{(5)}(J_x, J_y, J_z)$ singles out $J_y$ and is invariant under the permutation $J_x \leftrightarrow J_z$ and $P_z^{(5)}(J_x, J_y, J_z)$ singles out $J_z$ and is invariant under the permutation $J_x \leftrightarrow J_y$. One has similar results for the polynomial representation of the multiplication of the shift by $M = 6, 7, 9, 11,\ldots$. The explicit expressions of these polynomial representations are given in Appendix A.

Let us denote by $\Gamma_N$ a biquadratic corresponding to $u \rightarrow u \pm N \cdot \lambda$:

$$\Gamma_N(x, z) = (x^2 - 1) \cdot (z^2 - 1) \cdot J_x^{(N)} - (x^2 + 1) \cdot (z^2 + 1) \cdot J_y^{(N)} + 4 \cdot J_z^{(N)} \cdot x \cdot z = 0 \qquad (62)$$

In general, it should be noticed that the elimination of $y$ between $\Gamma_M(x, y)$ and $\Gamma_{M'}(y, z)$, yields a resultant which is factorized into $\Gamma_{(M+M')}(x, z)$ and $\Gamma_{(M-M')}(x, z)$ (for $M \geqslant M'$). When seeking for a new $\Gamma_N(x, z)$ there may be many $(M, M')$ enabling to get $\Gamma_N(x, z)$ (that is such that $N = M + M'$). One can verify that all these calculations give, as it should, the *same* result (in agreement with a polynomial representation of $u \rightarrow u \pm M \cdot \lambda \pm M' \cdot \lambda$ giving $u \pm (M + M') \cdot \lambda$ or $u \pm (M - M') \cdot \lambda$. Let us denote $T_N$ these homogeneous polynomial representations of the multiplication of the shift by the natural integer $N$. In the same spirit one can verify, for $N = M \cdot M'$ ($N$, $M$, $M'$ natural integers), that:

$$\begin{aligned} T_N(J_x, J_y, J_z) &= (T_M)^{M'}\,(J_x, J_y, J_z) = T_M(T_M(T_M(\,\cdots T_M(J_x, J_y, J_z) \cdots))) \\ &= (T_{M'})^{M}\,(J_x, J_y, J_z) \\ &= T_{M'}(T_{M'}(T_{M'}(\,\cdots T_{M'}(J_x, J_y, J_z) \cdots))) \end{aligned} \qquad (63)$$

One can, for instance, easily verify that $T_2$ and $T_3$ *commute*, as well as $T_2$ and $T_5$. Similarly one can verify, in a brute-force way, that $T_3$ and $T_5$ commute. This commutation relations are true for $T_N$ and $T_M$, for any $N$ and $M$. One thus has a *polynomial representation of the natural integers together with their multiplication*. One verifies easily that the homogeneous polynomial transformations $T_M$ are all of degree $M^2$, in $J_x$, $J_y$, $J_z$, for $M = 2, 3, 4, 5, 6, 7, 9, 11$.

### 6.1. Finite Order Conditions and Associated Algebraic Varieties

Let us show that one can deduce the (projective) finite order conditions $K^M(R) = \zeta \cdot R$, from the previous polynomial representations. Our motivation is that the corresponding algebraic varieties are "good candidates" for new free-(para?)-fermions, or new equivalent of the integrable chiral Potts model.[11] Actually, it will be shown, in a forthcoming publication, that the Baxterisation procedure *actually yields an elliptic foliation of the (general) anisotropic four state chiral Potts model.* Completely similar calculations can thus be performed, yielding an infinite set of "good candidates" for (higher genus) star-triangle integrability, enabling, in particular, to recover the higher genus integrable solution of Baxter–Perk–Au-Yang.[24] Of course, for the sixteen vertex model, we do not expect that one of this infinite set of finite order conditions could yield new Yang–Baxter integrable subcases of the sixteen vertex model.[19] We just consider the sixteen vertex model for heuristic reasons.

Since one knows that the (projective) *finite order conditions* of $\hat{K}^2$ often play a singled-out role for integrability (see the previous Section 3.1), and, in particular, since one knows[6] that the free-fermion conditions of the asymmetric eight vertex model correspond to $K^4(R) = \zeta \cdot R$, one can, as an exercise, try to systematically write, for the sixteen vertex model, the (projective) finite order conditions $K^{2N}(R) = \zeta \cdot R$, with $N$ natural integer.

Let us first give these finite order conditions for the Baxter model. For this heuristic model it is not necessary to explain, beyond the free-fermion subcase, the usefulness of the finite order conditions any further: these algebraic varieties correspond exactly to the set of RSOS models.[7]

At first sight, writing down the (projective) finite order condition $K^{2N}(R) = \zeta \cdot R$, corresponds to write four homogeneous equations on the four homogeneous parameters $a$, $b$, $c$ and $d$, yielding to points in the (projective) parameter space (codimension three). In fact the Baxter $R$-matrices of order two ($K^2(R) = \zeta \cdot R$) correspond to codimension two algebraic varieties. One easily check (see also Section 7.1) that $c = d = 0$ are such matrices. Furthermore, recalling[6] that the free-fermion condition for the $XYZ$ Hamiltonian, $J_z = 0$, corresponds to the finite order (projective) condition $K^4(R) = \zeta \cdot R$, one actually sees that the $R$-matrices of order four can actually correspond to a *codimension one* algebraic variety (only *one* algebraic condition on the homogeneous parameters of the model). Recalling the polynomial representation (55) of the shift doubling, one can easily get convinced that $J_z^{(2)} = 0$ should correspond to $K^8(R) = \zeta \cdot R$, also yielding *a only one* algebraic condition on the homogeneous parameters of the model (codimension one algebraic variety). This can be verified by a

[19] This (infinite) set of finite order conditions, corresponding to an infinite set of singled-out subcases of the sixteen vertex model should not be confused with the singled-out subcases obtained from the bifurcation analysis performed by Hijmans.[49] The finite order subcases correspond to the occurrence of an additional $Z_N$ discrete symmetry.

straight calculation. This is a general result: *all the finite order conditions* $K^{2N}(R) = \zeta \cdot R$ *correspond to codimension one algebraic varieties*, *expect* $N = 1$. The idea here is the following: $K$, or $\hat{K}$, corresponding, with some well-suited spectral parameter, to $\theta \to \theta + \eta$, $K^2$, or $\hat{K}^2$, must correspond to $\theta \to \theta + 2 \cdot \eta$. A finite order condition of order $M$ corresponds to a commensuration of $\eta$ with a period of the elliptic curves: $\eta = \mathcal{P}/M$, that is *just one* condition on the parameters of the model. Changing $K$ into $K^2$ amounts to changing $\eta$ into $2 \cdot \eta$, or equivalently, changing the order $M$ into $2M$. The fact that the finite order conditions $K^{2N}(R) = \zeta \cdot R$ correspond to *codimension one* algebraic varieties is thus a *consequence of the foliation* of the parameter space in elliptic curves.

More generally, a polynomial condition $C_{2N}(J_x, J_y, J_z) = 0$, corresponding to $K^{2N}(R) = \zeta \cdot R$, has to be compatible with the polynomial representations of $\lambda \to M \cdot \lambda$, for any integer $M$. This compatibility is often, in fact, an efficient way to get these finite order conditions. Explicit expressions of finite order conditions, as well as their compatibility with the shift doubling and, more generally, the polynomial representations of $\lambda \to M \cdot \lambda$, are given in Appendix B. For a prime integer $N \neq 2$ the algebraic varieties $P_x^{(N)}(J_x, J_y, J_z) = 0$, $P_y^{(N)}(J_x, J_y, J_z) = 0$, and $P_z^{(N)}(J_x, J_y, J_z) = 0$ give order $4N$ conditions:

$$K^{4N}(R) = \zeta \cdot R \tag{64}$$

Since the $P^{(N)}$'s (and the $J^{(N)}$'s for $N$ even) are functions of $J_x^2$, $J_y^2$ and $J_z^2$, the order $4N$ conditions, $K^{4N}(R) = \zeta \cdot R$, are also functions of the square $J_x^2$, $J_y^2$ and $J_z^2$. One can easily get infinite families of finite order conditions. For instance, iterating the shift doubling (55) (resp. (58)), and using this transformation on $J_z = 0$, one easily gets an *infinite number* of algebraic varieties corresponding to the finite order conditions of order $2^N$ (resp. $3^N$). Combining (55) and (58), one gets straightforwardly the finite order conditions of order $2^N \times 3^M$. More details will be given elsewhere.

**To sum up:** One sees that the, at first sight, "hardly Baxterisable" case of *finite order* iteration provides, to some extend, more results, and structures, than a "standard" *infinite order Baxterisation*: one gets, for instance, a *polynomial representation of the natural integers together with their multiplication*, this polynomial representation leaving invariant the *modular invariant*[54] of the elliptic curves, and giving *codimension-one* algebraic varieties compatible with this structure...

**Remark: Finite Order Conditions for the Sixteen Vertex Model.** Using the previous results, the decomposition (42) of the sixteen vertex model into a "$\hat{K}^2$-effective" Baxter model, and relation (43), one can obtain the finite order conditions, $K^{4N}(R) = \zeta \cdot R$, for the sixteen vertex model and find that they are actually *codimension-one* algebraic varieties. Recalling (43), one easily deduces that these finite order conditions are actually given in terms of the finite order conditions of the "$\hat{K}^2$-effective" Baxter model. These finite order conditions (see Appendix B) are simply expressed in terms of the associated "$\hat{K}^2$-effective" variables $J_x$, $J_y$ and $J_z$ which can be obtained from relations (44), (45). Eliminating the $J_x$, $J_y$ and $J_z$, one can write these finite order conditions (which are simple polynomial expressions of the $J_x$, $J_y$ and $J_z$) in terms of $\rho$, $\lambda$ and $\mu$ (see (44)). Using (20) one can write down the (homogeneous polynomial) expressions, corresponding to these finite order conditions, in terms of the sixteen homogeneous parameters of the sixteen vertex model. The calculations are straightforward but yield "huge" expressions in terms of these sixteen homogeneous parameters that will not be given here. However, one can easily check numerically these finite order conditions on $R$-matrices satisfying one of these finite order conditions, or even write some of these conditions for the asymmetric eight vertex model. For instance the $R$-matrix satisfying an order six condition of Appendix B:

$$R_{\text{sixteen}} = \begin{bmatrix} 1031472 & 1261594 & 1513016 & 1853022 \\ 1389212 & 1699970 & 2038844 & 2498102 \\ 2164474 & 2649426 & 3180102 & 3897318 \\ 2918010 & 3573322 & 4289126 & 5258478 \end{bmatrix} \tag{65}$$

actually verifies $\hat{K}^6(R_{\text{sixteen}}) = R_{\text{sixteen}}$. The asymmetric eight vertex model (7) corresponds to the following "effective" covariants $J_x$, $J_y$ and $J_z$:

$$\begin{gathered} Jx = ((aa'bb')^{1/2} + (cc'dd')^{1/2}), \qquad Jy = ((aa'bb')^{1/2} - (cc'dd')^{1/2}) \\ Jz = (aa' + bb' - cc' - dd')/2 \end{gathered} \tag{66}$$

Substituting (66) into the finite order conditions given in Appendix B, one gets several explicit examples of finite order conditions for the asymmetric six vertex model. One can easily verify (and understand) that these finite order conditions are (homogeneous) polynomial expressions (with integer coefficients) of the products $aa'$, $bb'$, $cc'$ and $dd'$ and that no square root occurs (see (66)).

## 7. LET US BAXTERISE QUANTUM HAMILTONIANS

Let us now consider a typical problem for any theoretician who wants to provide some "interesting contribution" in High-Tc superconductivity. Let us consider a strongly correlated quantum Hamiltonian which looks like a t-J model,[55–57] or a Hubbard model,[58, 59] or some coupled $XYZ$

quantum chains: how is it possible to see if this quantum Hamiltonian is integrable? More generally, let us consider a quantum Hamiltonian. How to see if it is possible to solve this quantum Hamiltonian? The Bethe Ansatz only works if one has some "conservation operator somewhere"[20] that enables to see this model as some "avatar" of the six-vertex model ($XXZ$ chain). Beyond this restricted Bethe Ansatz framework, and if one cannot, or does not know how to, associate a (commuting) family of transfer matrices, commuting with this Hamiltonian, one has very few tools left to solve this quantum Hamiltonian.[21] For a Yang–Baxter integrable model depending on one spectral parameter, the associated integrable quantum Hamiltonian can be seen as the derivative of the (logarithm) of the transfer matrix at a singled-out value of the spectral parameter: therefore the integrable quantum Hamiltonian *does not depend on the spectral parameter*. If the Bethe Ansatz is too complicated, or does not exist (higher genus curves,...), how can we make the spectral parameter(s) "emerge" so that the integrability structure becomes crystal clear? This is clearly another type of Baxterisation problem. Let us sketch how this can be done.

**Remark.** The derivation of the quantum Hamiltonian from a family of commuting transfer matrices is well-known, however the problem of the integrable deformations of a given integrable quantum Hamiltonian and the building of the (many parameters at first sight) family of transfer matrices, this extended quantum Hamiltonian commutes with, is a much more difficult problem. The next section tries to address these difficult problems, using already known simple examples. Since, in order to illustrate the Baxterisation of quantum Hamiltonians with simple exact analytical results, we revisit already known integrable examples, most of this section "looks like" the usual derivation of the hamiltonian from the transfer matrix, presented in reversed order. In fact, the Baxterisation of quantum Hamiltonians can be considered[22] *beyond* already known integrable cases, however the method becomes extremely difficult to handle in practice when the orbits of $\hat{K}^2$, with a given tangent corresponding to the Hamiltonian, are no longer (elliptic or rational) curves, but higher dimensional Abelian varieties (see the various figures in ref. 16), and,

[20] The typical example is the operator $\sum_n \sigma_n^z$ for the $XXZ$ Hamiltonian.(33, 60)

[21] The random matrix theory analysis of the level spacing distribution remains a possible tool to simply "detect" integrability.(61) However, besides the technical difficulties associated with the unfolding procedure, the calculations become very large for the $16 \times 16$ (Hubbard) $R$-matrices, or for two coupled spin chains.

[22] Numerically the Baxterisation of quantum Hamiltonians amounts to looking at the orbits (under the action of transformation $\hat{K}^2$) of points very close to the (singular) point corresponding to the permutation $P$ of the vertical and horizontal spaces (see below).

especially, when the singular point $P$ (see below) is *replaced by a higher dimensional singular locus* (see the remark in the next section). We just try here, modestly, to address some of the subtleties and difficulties arising in the Baxterisation of quantum Hamiltonians. One will, however, see that the Baxterisation procedure enables, in a first step, to get some hint on the $d$-dimensional vector spaces of the $R$-matrices (the "form" of the $R$-matrices) where the integrable algebraic subvarieties can live (see for instance (83), (84) and (90), in the following), and that integrability is associated with the occurrence of some additional factorization properties for the homogeneous transformation $K$.

## 7.1. Let Us Baxterise the *XYZ* Quantum Hamiltonian

Let us recall the $XYZ$ Hamiltonian,[36, 62] $H_{XYZ} = \sum_n H_{n,\,n+1}$, and let us represent $H_{n,\,n+1} = J_x \cdot \sigma_n^x \cdot \sigma_{n+1}^x + J_y \cdot \sigma_n^y \cdot \sigma_{n+1}^y + J_z \cdot \sigma_n^z \cdot \sigma_{n+1}^z$ and $P$, the matrix of permutation of the vertical space $n$ and $n+1$, as $4 \times 4$ matrices:

$$H_{n,\,n+1} = \begin{bmatrix} Jz & 0 & 0 & Jx - Jy \\ 0 & -Jz & Jx + Jy & 0 \\ 0 & Jx + Jy & -Jz & 0 \\ Jx - Jy & 0 & 0 & Jz \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{67}$$

Recalling Baxter's notations, the $R$-matrix, reads at order one in some small expansion parameter $\varepsilon$:

$$\begin{aligned} R &= P + \varepsilon \cdot P \cdot H_{n,\,n+1} + \cdots \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \varepsilon \cdot \begin{bmatrix} Jz & 0 & 0 & Jx - Jy \\ 0 & Jx + Jy & -Jz & 0 \\ 0 & -Jz & Jx + Jy & 0 \\ Jx - Jy & 0 & 0 & Jz \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + \varepsilon Jz & 0 & 0 & \varepsilon \cdot (Jx - Jy) \\ 0 & \varepsilon \cdot (Jx + Jy) & 1 - \varepsilon \cdot Jz & 0 \\ 0 & 1 - \varepsilon \cdot Jz & \varepsilon (Jx + Jy) & 0 \\ \varepsilon \cdot (Jx - Jy) & 0 & 0 & 1 + \varepsilon \cdot Jz \end{bmatrix} + \cdots \\ &= \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{bmatrix} \end{aligned} \tag{68}$$

One has the correspondence:

$$a = 1 + \varepsilon \cdot J_z + \cdots, \qquad b = \varepsilon \cdot (J_x + J_y) + \cdots$$
$$c = 1 - \varepsilon \cdot J_z + \cdots, \qquad d = \varepsilon \cdot (J_x - J_y) + \cdots$$

One verifies, immediately, that the algebraic $K$-covariants of the Baxter model read:

$$\frac{a^2 + b^2 - c^2 - d^2}{2} = 2\varepsilon \cdot J_z, \qquad a \cdot b + c \cdot d = 2\varepsilon \cdot J_x, \qquad a \cdot b - c \cdot d = 2\varepsilon \cdot J_y \tag{69}$$

the $\hat{K}$-invariants being (for instance) the ratio $J_x/J_z$ and $J_y/J_z$. The $R$-matrix $P$ can be seen to belong to *all* the elliptic curves of this foliation of the Baxter model (the invariants $(a^2 + b^2 - c^2 - d^2)/a/b$ or $ab/c/d$ are of the form 0/0). Matrix $P$ is really the equivalent of the *base point* of an elliptic foliation.[63]

Having the $XYZ$ Hamiltonian and willing to "recover" the Baxter $R$-matrix (and its canonical elliptic parameterization) is a slight modification of the previous Baxterisation process of $R$-matrices, or monodromy matrices, where we were building the algebraic variety from one arbitrary point of the algebraic variety. We do not have a point of the elliptic curve here, but rather a *singular point*, which is the equivalent of the *base point* of an elliptic foliation, namely point $P$, and a "vector" $(J_x, J_y, J_z)$ giving the *tangent* to the elliptic curve at point $P$. It is however clear that $[P, (J_x, J_y, J_z)]$ is *sufficient* to build, in a unique way, the elliptic curve, and thus the spectral parameter.

**A Heuristic Remark.** Point $P$ is a *singular point* of $K^2$ and the *diagonal* matrices are *fixed points* of $\hat{K}^2$:

$$K(P) = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \qquad K(K(P)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{K}^2 \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$$

More generally, considering the most general $4 \times 4$ matrices (sixteen vertex model), the singular matrices for $K^2$ are the $R$-matrices such that $K(R)$ is a rank two matrix. For the Baxter model singular matrices, such that $K^2(R)$ is the null matrix, are, for instance, $d=b=0$ or $a=c=0$. The *base points* of the elliptic foliation of the parameter space of the Baxter model are a subset of these singular subvarieties. The *base points* of the elliptic foliation are the points such that the $K$-invariante, $(a^2+b^2-c^2-d^2)/a/b$ or $ab/c/d$, are both of the form 0/0, namely $d=b=0$, $c=a$, or $d=b=0$, $a=-c$, or $a=d=0$, $b=c$, or $b=-c$, $a=d=0$. This can be easily generalized to the sixteen vertex model by imposing that the $p_i$'s $K$-covariants,[34] mentioned in Section 5.1, are all equal to zero.

In general these two sets of subvarieties, namely the singular subvarieties and the subvarieties of fixed points of $\hat{K}^2$, play a crucial role in the Baxterisation of a Hamiltonian. The subvariety of fixed points of $\hat{K}^2$ always contains the set of all the *diagonal* matrices. Not surprisingly we will see, in the next section, that diagonal matrices naturally occur (in a non-trivial way) in the Baxterisation process (see for instance (76) below).

## 7.2. Let Us Baxterise the t-J Quantum Hamiltonian

Let us recall the Hamiltonian of the t-J model,[55] and let us consider an ordering of the $9\times 9$ $R$-matrices, well-suited for the partial transposition $t_1$:

$$(+,+)(+,-)(+,0),(-,+),(-,-),(-,0),(0,+),(0,-)(0,0) \tag{70}$$

With this ordering, the "local Hamiltonian" of the t-J model, equivalent of the previous $H_{n,n+1}$, and the permutation matrix $P$, read respectively:

$$H_{n,n+1} = \begin{bmatrix} V+\frac{J}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V-\frac{J}{4} & 0 & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & \frac{J}{2} & 0 & V-\frac{J}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V+\frac{J}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{71}$$

The integrability cases[64] correspond to $(V, J) = (-t/2, 2t)$ or $(t/2, -2t)$ or $(3t/2, 2t)$ or $(-3t/2, -2t)$, The $R$-matrix reads at order one in some small expansion parameter $\varepsilon$:

$$R = P + \varepsilon \cdot P \cdot H_{n,n+1} + \cdots$$

$$= \begin{bmatrix} 1+\varepsilon\cdot(V+J/4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon\cdot J/2 & 0 & 1+\varepsilon\cdot(V-J/4) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon\cdot t & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1+\varepsilon\cdot(V-J/4) & 0 & \varepsilon\cdot J/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\varepsilon\cdot(V+J/4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon\cdot t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\varepsilon\cdot t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varepsilon\cdot t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \cdots \tag{72}$$

The (supersymmetric) integrable case $(V, J) = (-t/2, 2t)$ yields the (supersymmetric) "local Hamiltonian" $H_{n,n+1}(J = 2t, V = -t/2) = t \cdot H_{\text{susy}}$:

$$t \cdot H_{\text{susy}} = t \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{73}$$

Let us "absorb" the homogeneity of $H_{\text{susy}}$ by introducing $x = -\varepsilon \cdot t$. At first sight, one wants to Baxterise:

$$R_{\text{aw}} = P - x \cdot P \cdot H_{\text{susy}} + \cdots \tag{74}$$

Let us introduce two $9 \times 9$ matrices:

$$\hat{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\hat{N}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{75}$$

With ordering (70), matrix $\hat{N}$ is associated with the *electron-counting* operator which commutes with the t-J Hamiltonian. Matrix $\hat{N}$ commutes with $P$ and is such that the matrix $D = P \cdot H_{\text{susy}} - P + P \cdot \hat{N}$ is a *diagonal* matrix of successive diagonal entries: $1, 1, -1, 1, 1, -1, -1, -1, -1$. Actually, as far as the integrability of the model is concerned, one can add to the Hamiltonian an operator *which commutes with it*. Obviously one can also add to the $R$-matrix, a term $x \cdot P$ which just amounts to a global $1+x$ multiplicative factor in front of $P$. This amounts to introducing, instead of (74), the following $R$-matrix:

$$\begin{aligned} R_1(x) &= P - x \cdot \mathscr{D} = (1+x) \cdot P - x \cdot P \cdot H_{\text{susy}} - x \cdot P \cdot \hat{N} \\ &= (1+x) \cdot P \cdot \left( \mathscr{I} - \frac{x}{1+x} \cdot (H_{\text{susy}} + \hat{N}) \right) \\ &= \begin{bmatrix} 1-x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+x \end{bmatrix} \end{aligned} \tag{76}$$

where $\mathscr{I}$ denotes the $9 \times 9$ identity matrix. Under transformation $\hat{K}^4$ the $R$-matrix $R_1(x)$ becomes another matrix, of the *same form*,[23] but where $x \rightarrow x+2$:

$$\hat{K}^4(R_1(x)) = \frac{x-1}{x+1} \cdot R_1(x+2) \tag{77}$$

[23] Note that $\hat{K}^2(R_1)$ has a form similar to $R_1(x+1)$ up to *some change of signs* of some entries (supersymmetric graduation).

This solves the Baxterisation problem of (73), reducing it, after addition of "well-suited" commuting operators, to a *simple linear interpolation.*

• Another integrable case[(64)] for (71) is $(V, J) = (3t/2, 2t)$, for which one has to introduce *another correction* namely $\hat{N}'$ in (75), yielding the *same* $R$-matrix (76), namely $R_1 = P - x \cdot \mathcal{D}$, the diagonal matrix $\mathcal{D}$ being, now, equal to $P \cdot H_{tJ} - P + P \cdot \hat{N}'$, where $H_{tJ}$ is given by (71) for $(V, J) = (3t/2, 2t)$.

• Two other integrable cases[(64)] for (71) namely $(V, J) = (-3t/2, -2t)$ and $(V, J) = (t/2, -2t)$ yield the *same* $R$-matrix $R_2$. For $(V, J) = (-3t/2, -2t)$ and $(V, J) = (t/2, -2t)$ one has to introduce the correction $\hat{N}$ and $\hat{N}'$ respectively, yielding a very simple $R$-matrix $R_2 = x \cdot \mathcal{I} + P$. Similarly, under transformation $\hat{K}^2$, matrix $R_2(x)$ becomes another matrix of the *same form* but where $x \to x - 3$:

$$\hat{K}^2(R_2(x)) = \frac{x^2 - 1}{x \cdot (x-3)} \cdot R_2(x-3) \tag{78}$$

thus solving the Baxterisation problem of these last two integrable cases as a *simple linear interpolation* between $\mathcal{I}$ and $P$.

### 7.3. Hamiltonian Dependence of the Baxterisation Procedure

One certainly wants the Baxterisation procedure of a quantum Hamiltonian to be "universal:" it should be compatible with the addition of quantum operators which commute with the quantum Hamiltonian, and, thus, do not modify the integrability of the Hamiltonian.

The Baxterisation procedure is, of course, compatible with the *gauge transformations* on the quantum Hamiltonian:

$$H_{n,\,n+1} \to H_{n,\,n+1} + (G_n^\alpha \otimes \mathcal{I}_n - \mathcal{I}_{n+1} \otimes G_{n+1}^\alpha) \tag{79}$$

where $\mathcal{I}_n$ and $\mathcal{I}_{n+1}$ denote the identity operators on site $n$ and $n+1$ respectively, the "exponentiation" of (79) giving the $R \to g^{-1} \otimes h^{-1} \cdot R \cdot g \otimes h$ well-known gauge symmetries,[(44)] compatible with the Baxterisation: $\hat{K}^2(g^{-1} \otimes h^{-1} \cdot R \cdot g \otimes h) = g^{-1} \otimes h^{-1} \cdot \hat{K}^2(R) \cdot g \otimes h$. Furthermore, from the symmetries of the Baxterisation (see for instance (34)), one deduces that the Baxterisation of a quantum Hamiltonian is necessarily compatible with the following deformations of $H_{n,\,n,\,+1}$ generalizing[24] (79):

$$H_{n,\,n+1} \to H_{n,\,n+1} + (G_n^\alpha \otimes \mathcal{I}_n - \mathcal{I}_{n+1} \otimes G_{n+1}^\alpha) + (\tilde{G}_n^\beta \otimes \mathcal{I}_n + \mathcal{I}_{n+1} \otimes \tilde{G}_{n+1}^\beta) \tag{80}$$

[24] A simple example corresponds, for instance, to adding operator $\sum_n \sigma_n^z$ to the *XXZ* Hamiltonian.[(33, 60)]

The Baxterisation should also provide (at least as far as integrability is concerned) compatible results when one adds, to the quantum Hamiltonian $\sum_n H_{n,\,n+1}$, operators commuting with it:

$$H_{n,\,n+1} \rightarrow H_{n,\,n+1} + O^{\alpha}_{n,\,n+1}, \qquad \alpha = 1,..., r \tag{81}$$

where the $\sum_n O^{\alpha}_{n,\,n+1}$'s commute with $\sum_n H_{n,\,n+1}$. When the quantum Hamiltonian $\sum_n H_{n,\,n+1}$, is integrable, one would like to have a complete description of *all* the *integrable deformations* (81), and a complete description of the algebraic variety associated with the Baxterisation of the largest integrable deformation of the quantum Hamiltonian (as many $O^{\alpha}_{n,\,n+1}$'s as possible). Are all the integrable deformations $O^{\alpha}_{n,\,n+1}$ necessarily of the $\hat{K}^2$-compatible form (80)? All the $O^{\alpha}_{n,\,n+1}$'s are such that the $\sum_n O^{\alpha}_{n,\,n+1}$'s commute with $\sum_n H_{n,\,n+1}$, but do all the $O^{\alpha}_{n,\,n+1}$'s commute with $H_{n,\,n+1}$ and commute[25] all together? Is there a way to find $r$, the largest number of integrable deformations $O^{\alpha}_{n,\,n+1}$, by simple arguments? Is it possible to find lower and upper bounds[26] for $r$? All these questions will be analysed in details elsewhere. Let us just give, in the following sections, some partial answers to these questions based on deformations of the two previous "heuristic" integrable cases, namely $R$-matrices $R_1(x)$ and $R_2(x)$.

Let us try to see what happens when one changes the integrable Hamiltonian adding an operator which commutes with it. Suppose that, instead of introducing the $R$-matrix $R_1(x)$, which yields a simple linear interpolation for the Baxterisation process (76), one introduces:

$$\begin{aligned} R(x, y) &= P - x \cdot \mathscr{D} + y \cdot P \cdot \hat{N} \\ &= (1+x) \cdot P - x \cdot P \cdot H_{\mathrm{susy}} - (x-y) \cdot P \cdot \hat{N} \end{aligned} \tag{82}$$

One can actually easily verify that the $n$th $\hat{K}^2$-iterate of $R(x, y)$ are all linear combinations of $R(x, y)$, $\hat{K}^2(R(x, y))$, $\hat{K}^4(R(x, y))$, $\hat{K}^6(R(x, y))$ and $\hat{K}^8(R(x, y))$ or, equivalently, $R(x, y)$, $\hat{K}^2(R(x, y))$, $\hat{K}^4(R(x, y))$, $P$ and $P \cdot \hat{N}$:

[25] In the case where $\hat{K}^2$ is an infinite order transformation densifying an algebraic variety, this variety is an Abelian variety and one can deduce, from this Abelian property, that the $P \cdot O^{\alpha}_{n,\,n+1}$'s should commute.

[26] For Abelian varieties parameterized by theta functions of $g$ variables, one could imagine that $r$ should be related to $g + N_g$, where $N_g$ denotes the number of independent gauge deformations (79).

$$
\begin{aligned}
\hat{K}^{2n}(R(x, y)) &= A_0^{(n)} \cdot R(x, y) + A_1^{(n)} \cdot \hat{K}^2(R(x, y)) \\
&\quad + A_2^{(n)} \cdot \hat{K}^4(R(x, y)) + A_3^{(n)} \cdot P + A_4^{(n)} \cdot P \cdot \hat{N} \\
&= B_0^{(n)} \cdot R(x, y) + B_1^{(n)} \cdot \hat{K}^2(R(x, y)) + B_2^{(n)} \cdot \hat{K}^4(R(x, y)) \\
&\quad + B_3^{(n)} \cdot \hat{K}^6(R(x, y)) + B_4^{(n)} \cdot \hat{K}^8(R(x, y))
\end{aligned} \tag{83}
$$

In other words, the Baxterisation acts in the four-dimensional vector space[27] of the $R$-matrices of the form (83). The Baxterisation of $R(x, y)$ gives successive $9 \times 9$ matrices $\hat{K}^{2n}(R(x, y))$ of the form:

$$
R(A, B, C, D, E, F) = \begin{bmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B & 0 & C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D & 0 & 0 & 0 & E & 0 & 0 \\
0 & C & 0 & B & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D & 0 & E & 0 \\
0 & 0 & E & 0 & 0 & 0 & D & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E & 0 & D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F
\end{bmatrix} \quad \text{where} \quad A = B + C \tag{84}
$$

**Factorization scheme:** Let us Baxterise $m_0 = R(x, y)$ using the *homogeneous* transformation $K$. The first iteration reads $m_1 = K(m_0)$. Let us denote $G_1$ the gcd of *all the entries* of matrix $m_1$. In order to go on iterating, it is better to introduce the "reduced" matrix, $M_1 = m_1/G_1$. Let us denote $m_2 = K(M_1)$. Similarly, one can introduce $G_2$ the gcd of all the entries of matrix $m_2$, and define the "reduced" matrix, $M_2 = m_2/G_2$, and so on. Introducing $G(w)$ the generating function of the degree of the successive gcd's $G_n$'s, and $D(w)$ the generating function of the degree of the determinants of the successive reduced matrices $M_n$'s, one gets for the iteration of (for instance) $R(x, 2)$:

$$
\begin{aligned}
G(w) &= 4w + 25w^2 + 34w^3 + 144w^4 + 164w^5 + 604w^6 + 684w^7 + \, + 2444w^8 \\
&\quad + 2764w^9 \cdots \\
D(w) &= 9 + 36w + 63w^2 + 198w^3 + 288w^4 + 828w^5 + 1188w^6 + 3348w^7 \\
&\quad + 4788w^8 + 13428w^9 + \cdots
\end{aligned} \tag{85}
$$

[27] Five homogeneous parameters. Note that the odd $\hat{K}$-iterate do not belong to this four-dimensional vector space but to its transformed under $t_1$.

Of course one easily verifies that $(1-8w)\cdot D(w)+9G(w)-9=0$, which is simply deduced from:

$$M_{n+1}=\frac{K(M_n)}{G_{n+1}} \tag{86}$$

One easily verifies, up to order nine, that (85) are the expansions of the two *rational* expressions:

$$\begin{aligned} G(w) &= \frac{w\cdot(4+21w-7w^2+26w^3-16w^4)}{(1-w)(1+2w)(1-2w)} \\ D(w) &= 9\cdot\frac{(1+3w-2w^2)(1+w^2)}{(1-w)(1+2w)(1-2w)} \end{aligned} \tag{87}$$

There is nothing specific with $y=2$: beyond $R(x, 2)$, one also gets (generically) a $2^n$ *exponential growth* of the iteration calculations for $R(x, y)$, thus excluding Yang–Baxter integrability. This may, at first sight, seem in contradiction with the fact that the (local) Hamiltonian $H=H_{\text{susy}}+\rho\cdot\hat{N}$ is "as integrable as" $H_{\text{susy}}$ (or $H_{\text{susy}}+\hat{N}$). In fact the (local) Hamiltonian $H=H_{\text{susy}}+\rho\cdot\hat{N}$ should also yield a *polynomial growth* of the calculation corresponding to some algebraic subvariety of (84) such that the *tangent space* to this algebraic subvariety, at point $R=P$, contains the vector $P\cdot H=P\cdot(H_{\text{susy}}+\rho\cdot\hat{N})$. In other words, $R(x, y)$ should only be integrable in the $(x, y)\to(0, 0)$ limit, but *not for finite values* for $(x, y)$. In our case the plane containing point $P$ and the two integrable "vectors" $P\cdot H_{\text{susy}}$ and $P\cdot(H_{\text{susy}}+\rho\cdot\hat{N})$, is not included in the algebraic subvariety of polynomial growth. This is a quite general situation: in general, integrability does not correspond to *linear spaces* but to *algebraic subvarieties* with some "curvature."

Deforming (76) with $P\cdot\hat{N}$, one thus finds that the integrability of (76) is "surrounded" by a (four-dimensional) space corresponding, generically, to a $2^n$ *exponential growth*. However this *does not mean* that the *whole* four-dimensional space (84) (except (76) of course) corresponds to a $2^n$ exponential growth: it is possible that some algebraic *subvarieties* of (84) could correspond to a *polynomial growth* of the iteration calculations and, possibly, correspond to a Yang–Baxter integrability. The question is thus to find such polynomial growth, and possibly Yang–Baxter integrable, algebraic subvarieties.[28] The next section gives an explicit example of

[28] This should be particularly difficult when these subvarieties are not codimension-one subvarieties, but codimension two, three (or more...) subvarieties.

particular Yang–Baxter integrable deformations of the simple linear interpolation $R_2 = \mathcal{I} + x \cdot P$, and gives the associated integrable algebraic subvariety. Section 7.5 sketches a strategy to find systematically[29] these polynomial growth, and integrable, subvarieties (or, more generally, the subvarieties of smaller topological entropy[25, 26]).

### 7.4. Beyond Simple Linear Interpolation: An Integrable Deformation of $R_2 = \mathcal{I} + x \cdot P$

Matrix $R_2 = \mathcal{I} + x \cdot P$ gives a Baxterisation of some integrable t-J Hamiltonian as a simple *linear interpolation* between $P$ and the identity matrix $\mathcal{I}$. Let us try to extend this simple linear interpolation, and "merge" it into a larger integrable $R$-matrix. Similarly to what has been previously performed, let us consider two matrices $\mathcal{N}$ and $\mathcal{A}$, which respectively commute, and *anticommute*, with $P$:

$$\mathcal{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{88}$$

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{89}$$

[29] Of course one can always iterate *numerically* a point, very close to point $R = P$, moving away from this point along directions like (for $R_1$) $P \cdot H_{\rm susy}$, $P \cdot \hat{N}$ or $P \cdot \hat{N}'$ and visualize these integrable subvarieties.[16]

Do note that $P \cdot \mathscr{A}$ is a *diagonal* matrix with diagonal entries $0, -1, 1, 1, 0, -1, -1, 1, 0$. Let us consider a deformation of $R_2 = \mathscr{I} + x \cdot P$ with these two matrices. This amounts to considering $R = P + \alpha \cdot \mathscr{I} + \beta \cdot \mathscr{A} + \gamma \cdot \mathscr{N}$, or equivalently the non-symmetric $R$-matrix:

$$R_z(t, z) = \begin{bmatrix} D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 & C & 0 & 0 \\ 0 & C & 0 & A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & B & 0 \\ 0 & 0 & B & 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D \end{bmatrix} \tag{90}$$

This form (90) is stable by $\hat{K}^2$. The $P + \alpha \cdot \mathscr{I} + \beta \cdot \mathscr{A} + \gamma \cdot \mathscr{N}$ algebra is thus an algebra well-suited for the action of $\hat{K}^2$. Let us consider (90) for:

$$A = t \cdot (1 - z^6), \qquad B = z^4 \cdot (t^2 - 1), \qquad C = z^2 \cdot (t^2 - 1), \qquad D = t^2 - z^6 \tag{91}$$

The corresponding $R$-matrix ((90), (91)) Baxterises in a very simple way:

$$\beta(t, z) \cdot \hat{K}^2(R_z(t, z)) = R_z(t, t \cdot z)$$
$$\text{where} \quad \beta(t, z) = \frac{(1 - z^3)(1 + z^3)(z^3 t^3 - 1)(z^3 t^3 + 1)\, t^2}{(t + z^3)(z^3 t + 1)(z^3 t - 1)(t - z^3)} \tag{92}$$

The $R$-matrix (90), with (91), is actually one of the Yang–Baxter integrable models of Perk and Schultz,[65, 66] and can be seen as a deformation of $R_2(x)$. Actually, in the limit $(z, t) = (1 + Z \cdot h, 1 + T \cdot h)$, where $h \to 0$, one finds that operator $\mathscr{A}$ occurs at order one in $h$, while operator $\mathscr{N}$ occurs at order two in $h$:

$$\frac{2 R_z(t, z)}{z^2 \cdot (z^2 + 1) \cdot (t^2 - 1)}$$
$$\to \left( P - 3\, \frac{Z}{T} \cdot \mathscr{I} \right) + \left( Z \cdot \mathscr{A} + 3\, \frac{Z(Z - T)}{2T} \cdot \mathscr{I} \right) \cdot h$$
$$+ \left( \frac{Z \cdot (8Z - 3T)}{2} \cdot \mathscr{N} - \frac{Z^2}{2} \cdot \mathscr{A} - \frac{Z(16Z^2 - 3TZ - 3T^2)}{4T} \cdot \mathscr{I} \right) \cdot h^2 + \cdots$$

Parametrization (91) is equivalent to the (Yang–Baxter) integrability[65, 66] condition:

$$\mathscr{C}(A, B, C, D) = B^3D - B^2C^2 + C^3D + BCA^2 - BCD^2 = 0 \tag{93}$$

This algebraic variety is an extension of the simple interpolation between the identity matrix and $P$ (Yang $R$-matrix), which corresponds to $C = B$ and $D = A + B$. Actually, for $C = B$, condition (93) factorizes as follows:

$$\mathscr{C}(A, B, B, D) = -B^2 \cdot (D - (B - A)) \cdot (D - (A + B)) = 0 \tag{94}$$

Considering the general $R$-matrix of the form (90) (*beyond* the integrability condition (91)), the factorization analysis, which extracts the gcd's at every step (see (86)), yields a $2^n$ *exponential growth* of the calculations. The generating functions of the degrees of the $M_n$'s and $G_n$'s (see (86), (85)) read (the expansions of $D(w)$ and $G(w)$ have been obtained up to $w^8$):

$$\begin{aligned} D(w) &= 3 \cdot \frac{3 - 2w + 6w^2 + 2w^3 - 6w^4}{(1-w)(1+w)(1-2w)} \\ G(w) &= w \cdot \frac{6 - 7w + 18w^2 + 6w^3 - 16w^4}{(1-w)(1+w)(1-2w)} \end{aligned} \tag{95}$$

In order to have a *polynomial growth* of the calculations some *additional factorizations must occur*. Actually, for (91), that is when the integrability condition (93) is satisfied, the generating functions of the degrees[30] now read (the expansions have been obtained up to $w^8$):

$$D(w) = 3 \cdot \frac{6 - 2w + 7w^2 + 7w^3}{(1-w^2)(1-w)}, \qquad G(w) = w \cdot \frac{(14 - 9w + 19w^2 + 18w^3)}{(1-w^2)(1-w)} \tag{96}$$

## 7.5. Integrability Emerging from the Occurrence of Additional Factorizations

In fact one can actually find the integrability condition (93), as a *condition corresponding to the occurrence of additional factorizations*. The "bifurcation" from the polynomial growth of the degrees to a $2^n$ exponential growth of the degrees, takes place immediately when leaving the integrability condition (93). In the $(A, B, C, D)$ parameter space, the

[30] The degree, here, being the degree in $t$, the calculations being performed for (91) with $z = 3$. Note that there is nothing specific with $z = 3$.

integrability condition (93) is "surrounded" by a $2^n$ exponential growth. More specifically, the "bifurcation" from the polynomial growth of the degrees to a $2^n$ exponential growth of the degrees, takes place *with the first gcd*, namely $G_1$. Let us consider the factorization of the $R$-matrix corresponding to (91) (that is (93)), and the one where $A$ has been replaced by $A+u$, which corresponds to a $2^n$ exponential growth. The first gcd's corresponding to these two situations, namely for $u=0$ (polynomial growth, integrable) $g_1$, and for $u \neq 0$ ($2^n$ exponential growth) $G_1(u)$, read respectively:[31]

$$\begin{aligned} G_1(u) &= (-u+z^3-t+tz^6-z^3t^2)^2 \cdot (-u-z^3-t+tz^6+z^3t^2)^2 \\ &\quad \cdot (t^2-z^6)^2 = D^2 \cdot (B \cdot C - A^2)^2 \\ g_1 &= (t^2-z^6)^5 \cdot (z^6t^2-1)^2 \end{aligned} \tag{97}$$

One remarks that:

$$g_1 = (t^2-z^6) \cdot G_1(u=0) = D \cdot G_1(u=0) \tag{98}$$

which amounts to saying that, restricting to the integrability condition (93) (here $u=0$), an *additional factorization occurs*, changing (95), and its $2^N$ exponential growth of the calculations, into (96) and its associated *polynomial growth*. When the integrability condition (93) is not satisfied, one has $G_1 = D^2 \cdot (B \cdot C - A^2)^2$, and the first "reduced" matrix $M_1$ reads:

$$M_1 = \frac{K(R)}{G_1}$$
$$= \begin{bmatrix} A^2-BC & 0 & 0 & 0 & -DC & 0 & 0 & 0 & -DB \\ 0 & DA & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & DA & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & DA & 0 & 0 & 0 & 0 & 0 \\ -DB & 0 & 0 & 0 & A^2-BC & 0 & 0 & 0 & -DC \\ 0 & 0 & 0 & 0 & 0 & DA & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & DA & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & DA & 0 \\ -DC & 0 & 0 & 0 & -DB & 0 & 0 & 0 & A^2-BC \end{bmatrix} \tag{99}$$

[31] Note that the factors of $G_1(u)$ actually correspond to some factors in $\det(R(u))$ which reads $\det(R(u)) = (-u+z^3-t+tz^6-z^3t^2)^3 \cdot (-u-z^3-t+tz^6+z^3t^2)^3 \cdot (t^2-z^6)^3$. This is a general result: the cofactors associated with $K^n$ or $t_1$ and $I$ are necessarily related with powers of the factors occurring in the determinant (or the whole determinant), and their transformed by the group generated by $t_1$ and $I$.

Condition (93), namely $(B^3 + C^3)\, D + ((A^2 - BC) - D^2)\, BC = 0$, is *nothing but the condition of divisibility* of $A^2 - BC$ by $D$, which obviously enables to extract an *additional factor* $D$ in all the entries of (99). The fact that (93) is such a condition of divisibility, becomes obvious if one uses the fact that (93) is a *rational surface*, and one actually uses this rational parameterization. With parameterization (91) the division of $A^2 - BC$ by $D$ reads:

$$\frac{A^2 - BC}{D} = 1 - z^6 \cdot t^2 \tag{100}$$

Finding systematically algebraic varieties (like (93) but not necessarily rational) which correspond to *additional factorizations* in the factorization scheme, and thus correspond to a smaller complexity of the calculations (smaller Arnold complexity,[25] smaller topological entropy[26]), *can be seen as a new approach* of integrable models. The set of all these singled-out algebraic varieties, of non generic topological entropy,[25] should be a *complicated stratified space*. A remarkable subset is the set of algebraic varieties associated with a *polynomial growth* of the calculations. A remarkable subset of the previous subset is the set of algebraic varieties which are Yang–Baxter integrable.

In this *integrable deformations* framework (81) one notes that the $9 \times 9$ matrices $H_{n,\, n+1}$, considered for different values of their parameters $V$, $J$, $t$, commute all together and commute with $P$ and $P \cdot H_{n,\, n+1}$. Therefore one could imagine that the matrices $R_1$, and $R_2$, could belong to the *same* integrable algebraic variety. Is it possible to give a complete description of *all* the *integrable deformations* (81), and a complete description of the algebraic varieties associated with the Baxterisation of the largest integrable deformation of such quantum Hamiltonians? These questions will be studied elsewhere.

## 8. LET US BAXTERISE DIFFERENTIAL OPERATORS

The Baxterisation is the building of the spectral parameters wherever they live: on an elliptic curve, on a rational surface, on a Jacobian variety, on an Abelian variety,... The Baxterisation of $R$-matrices, or monodromy matrices, can always be performed systematically using various methods (analytical methods, by formal calculations, by visualization of the orbits,[15, 16]...). Let us show, in this section, that these results can even be extended to *infinite dimensional* generalizations of monodromy matrices, namely when the block matrices in the $R$-matrix are infinite dimensional linear operators ($L$-operators, local quantum Lax matrices...).

Let us first recall the Toda chain.[67, 68] The model represents a chain of point particles described by coordinates $q_n$ and momentum $p_m$ with canonical Poisson brackets $\{p_m, q_n\} = \delta_{m,n}$ (in the classical case) and $[p_m, q_n] = -i\delta_{m,n}$ (in the quantum case). The Hamiltonian and the $L$-operator are respectively, *both in the classical and quantum case*[32] (see for instance, ref. 71 or page 197 in ref. 35):

$$H = \sum_n \left( \frac{p_n^2}{2} + e^{q_{n+1} - q_n} \right), \qquad L[\lambda] = \begin{bmatrix} \lambda - p_n & e^{q_n} \\ e^{-q_n} & 0 \end{bmatrix} \tag{101}$$

Let us consider, here, a $L$-operator similar to the one of the quantum Toda chain:

$$L[\lambda] = \begin{bmatrix} \lambda - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} \tag{102}$$

This can be seen as a generalization of the monodromy matrices (23) where the $m \times m$ matrices $A$, $B$, $C$ and $D$ become operators which can only have *infinite dimensional representations*. Let us introduce an "inverse" $L$-operator:

$$I(L)[\lambda'] = \begin{bmatrix} 0 & e^x \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} \tag{103}$$

and let us perform the products of these two $L$-operators:

$$L[\lambda] \cdot I(L)[\lambda'] = \begin{bmatrix} \lambda - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & e^x \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}$$

$$\text{where} \quad N = \left( \lambda - \frac{d}{dx} \right) \cdot e^x + e^x \cdot \left( \lambda' + \frac{d}{dx} \right) \tag{104}$$

[32] The inverse scattering method on this model has been investigated in refs. 69 and 70. This enables to introduce (see for instance page 197 in ref. 35) an auxiliary problem associated with the $2 \times 2$ $L$-operator (101). For more details on the $L$-operators (or local transition matrix) and the relations with inverse scattering (Lax pairs...) see for instance refs. 73 and 74, or pages from 187 to 235 in ref. 35.

or:

$$I(L)[\lambda'] \cdot L[\lambda] = \begin{bmatrix} 0 & e^{x} \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} \cdot \begin{bmatrix} \lambda - \dfrac{d}{dx} & e^{x} \\ e^{-x} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ N' & 1 \end{bmatrix}$$

$$\text{where} \quad N' = e^{-x} \cdot \left(\lambda - \frac{d}{dx}\right) + \left(\lambda' + \frac{d}{dx}\right) \cdot e^{-x} \tag{105}$$

It is a straightforward calculation to see that the derivative operator $d/dx$ disappears in the off-diagonal $N$ and $N'$ operators, and that $N$ and $N'$ reduce to the operators of multiplication by the function $(\lambda' + \lambda - 1) \cdot e^{x}$, and $(\lambda' + \lambda - 1) \cdot e^{-x}$, respectively. The operators $N$ and $N'$ reduce to the null operator for $\lambda' = 1 - \lambda$. Therefore one has the following result:

$$L[\lambda] \cdot I(L)[1-\lambda] = I(L)[1-\lambda] \cdot L[\lambda] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{106}$$

As it should the left "inverse" coincides with the right "inverse." Let us use such inverse and combine it with the partial transposition $t_1$, previously described. Let us now perform the partial transposition $t_1$ which amounts to permuting the two off-diagonal operators:

$$t_1(L)[\lambda] = \begin{bmatrix} \lambda - \dfrac{d}{dx} & e^{-x} \\ e^{x} & 0 \end{bmatrix} \tag{107}$$

One easily finds, as a consequence of the nullity of two operators $N_1$ and $N_2$, when $\lambda' = -1 - \lambda$:

$$N_1 = \left(\lambda - \frac{d}{dx}\right) \cdot e^{-x} + e^{-x} \cdot \left(\lambda' + \frac{d}{dx}\right) \qquad \text{and}$$

$$N_2 = e^{x} \cdot \left(\lambda - \frac{d}{dx}\right) + \left(\lambda' + \frac{d}{dx}\right) \cdot e^{x}$$

that the inverse of (107) reads:

$$I(t_1(L))[\lambda'] = \begin{bmatrix} 0 & e^{-x} \\ e^{x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} \qquad \text{where} \quad \lambda' = -1 - \lambda \tag{108}$$

Let us perform, again, the partial transposition $t_1$ on the previous $L$-operator:

$$t_1(I(t_1(L)))[\lambda'] = \begin{bmatrix} 0 & e^x \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} \tag{109}$$

The inverse of (109) reads:

$$I(t_1(I(t_1(L))))[\lambda''] = \begin{bmatrix} \lambda'' - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} \tag{110}$$

with $\lambda'' = -\lambda' + 1$. Actually:

$$\begin{aligned} & t_1(I(t_1(L)))[\lambda'] \cdot I(t_1(I(t_1(L))))[\lambda''] \\ & \quad = \begin{bmatrix} 0 & e^x \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} \cdot \begin{bmatrix} \lambda'' - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ N_3 & 1 \end{bmatrix} \\ & I(t_1(I(t_1(L))))[\lambda''] \cdot t_1(I(t_1(L)))[\lambda'] \\ & \quad = \begin{bmatrix} \lambda'' - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & e^x \\ e^{-x} & \lambda' + \dfrac{d}{dx} \end{bmatrix} = \begin{bmatrix} 1 & N_4 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{111}$$

where:

$$\begin{aligned} N_3 &= e^{-x} \cdot \left(\lambda'' - \frac{d}{dx}\right) + \left(\lambda' + \frac{d}{dx}\right) \cdot e^{-x} = (\lambda'' + \lambda' - 1) \cdot e^{-x} \qquad \text{and} \\ N_4 &= \left(\lambda'' - \frac{d}{dx}\right) \cdot e^x + e^x \cdot \left(\lambda' + \frac{d}{dx}\right) = (\lambda'' + \lambda' - 1) \cdot e^x \end{aligned} \tag{112}$$

Combining this result, and the previous result (108), one gets that $I(t_1(I(t_1(L))))$ *has exactly the same form as L, but with* $\lambda$ *changed into* $\lambda + 2$:

$$I(t_1(I(t_1(L))))[\lambda] = L[\lambda + 2] = \begin{bmatrix} (\lambda + 2) - \dfrac{d}{dx} & e^x \\ e^{-x} & 0 \end{bmatrix} \tag{113}$$

$It_1 It_1$ is an *infinite order* transformation acting on the $L$-operator (102). This provides a first, and very simple, example of *rational* Baxterisation of the $L$-operator (102). The iteration of $It_1 It_1$ yields a "trajectory," in the space of the $L$-operator (102), corresponding to a *straight line*: $\lambda \to \lambda + 2$. It seems necessary, here, to perform $(It_1)^2$, instead of $It_1$, to preserve the form of the $L$-operator. In fact one can only perform $It_1$ if one recognizes that (108) is like $L(\lambda+1)$, up to a transformation $c \cdot s$ which commutes with the inversion $I$ and with the partial transposition $t_1$:

$$\begin{bmatrix} 0 & e^{-x} \\ e^{x} & -1-\lambda+\dfrac{d}{dx} \end{bmatrix} = c\left( s\left( \begin{bmatrix} \lambda+1-\dfrac{d}{dx} & e^{x} \\ e^{-x} & 0 \end{bmatrix} \right)\right) \tag{114}$$

where $c$ and $s$ are the two following transformations ($U$, $V$, $W$, $T$ are any operator):

$$c\left(\begin{bmatrix} U & V \\ W & T \end{bmatrix}\right) = \begin{bmatrix} T & W \\ V & U \end{bmatrix} \quad \text{and} \quad s\left(\begin{bmatrix} U & V \\ W & T \end{bmatrix}\right) = \begin{bmatrix} -U & V \\ W & T \end{bmatrix} \tag{115}$$

Therefore $It_1$ is associated with the shift $\lambda \to \lambda + 1$. One sees that $\lambda$ actually *plays the role of the spectral parameter* for the Toda $L$-operator.

Many other examples, corresponding to differential operators and $L$-operators, or local quantum Lax matrices associated with the $XXX$ quantum Hamiltonian, yield similar calculations: for instance, a very simple example of $L$-operator, associated with the discrete self-trapping model (see for instance ref. 72) can easily be Baxterised, the calculations being extremely similar to the one sketched in this section.

## 8.1. Let Us Baxterise Higher Derivatives

Let us consider a (slightly) more complicated example of Baxterisation of differential operators. In the following we will denote by $p$ the derivative operator $d/dx$. More generally, let us consider the following $L$-operator:

$$L_N = \begin{bmatrix} \mathscr{A}^{(0)}(x) & B(x) \\ C(x) & 0 \end{bmatrix} \tag{116}$$

where $\mathscr{A}^{(0)}(x)$ denotes the differential operator $A_0(x) + A_1(x) \cdot p + \cdots + A_m(x) \cdot p^m + \cdots + A_N(x) \cdot p^N$ and where the two functions $B(x)$ and $C(x)$

are related by $B(x) \cdot C(x) = 1$. Let us first perform the partial transposition $t_1$:

$$t_1(L_N) = \begin{bmatrix} A_0(x) + A_1(x) \cdot p + \cdots + A_m(x) \cdot p^m + \cdots + A_N(x) \cdot p^N & C(x) \\ B(x) & 0 \end{bmatrix} \tag{117}$$

Let us try to find the inverse of this $L$-operator. Let us denote $\mathscr{A}^{(1)}(x)$ another similar differential operator $\tilde{A}_0(x) + \tilde{A}_1(x) \cdot p + \cdots + \tilde{A}_m(x) \cdot p^m + \cdots + \tilde{A}_N(x) \cdot p^N$. It is straightforward to see that $t_1(L_N) \cdot I(t_1(L_N))$ is the identity matrix when $I(t_1(L_N))$ reads:

$$I(t_1(L_N)) = \begin{bmatrix} 0 & C(x) \\ B(x) & -\mathscr{A}^{(1)}(x) \end{bmatrix} \tag{118}$$

with:

$$\mathscr{A}^{(0)}(x) \cdot C(x) = C(x) \cdot \mathscr{A}^{(1)}(x) \tag{119}$$

The fact that the left inverse and the right inverse identify is just a consequence of the fact that (119) is equivalent to $B(x) \cdot \mathscr{A}^{(0)}(x) = \mathscr{A}^{(1)}(x) \cdot B(x)$ when $B(x) \cdot C(x) = 1$. One easily sees that, up to the previous $c$ and $s$ transformations, the infinite order transformation $It_1$ reads:

$$I\left(t_1\left(\begin{bmatrix} \mathscr{A}^{(0)}(x) & B(x) \\ C(x) & 0 \end{bmatrix}\right)\right) = c\left(s\left(\begin{bmatrix} \mathscr{A}^{(1)}(x) & B(x) \\ C(x) & 0 \end{bmatrix}\right)\right) \tag{120}$$

where $\mathscr{A}^{(1)}(x)$ is deduced from $\mathscr{A}^{(0)}(x)$ by (119). From now on let us note $\tilde{A}_N$, $A_N$, $C$ the functions $\tilde{A}_N(x)$, $A_N(x)$ and $C(x)$. We will denote $C'$, $C''$ and $C^{(N)}$ the first, second and $N$-th derivatives of $C(x)$. For $N = 3$ the transformation $\mathscr{A}^{(0)}(x) \rightarrow \mathscr{A}^{(1)}(x)$ reads:

$$\begin{aligned} C \cdot \tilde{A}_3 &= C \cdot A_3 \\ C \cdot \tilde{A}_2 &= C \cdot A_2 + 3 \cdot C' \cdot A_3 \\ C \cdot \tilde{A}_1 &= C \cdot A_1 + 2 \cdot C' \cdot A_2 + 3 \cdot C'' \cdot A_3 \\ C \cdot \tilde{A}_0 &= C \cdot A_0 + \cdot C' \cdot A_1 + C'' \cdot A_2 + C^{(3)} \cdot A_3 \end{aligned} \tag{121}$$

Let us denote $S_1 = C'/C$, $S_2 = C''/C$ and $S_3 = C^{(3)}/C$. One can straightforwardly associate to transformation (121) a $4 \times 4$ matrix:

$$M = \begin{bmatrix} 1 & S_1 & S_2 & S_3 \\ 0 & 1 & 2S_1 & 3S_2 \\ 0 & 0 & 1 & 3S_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{122}$$

The Baxterisation process corresponds to the iteration of this matrix. Note that this iteration yields a simple *group structure*:

$$M^n = \begin{bmatrix} 1 & n\cdot S_1 & n\cdot S_2+n\cdot(n-1)\cdot {S_1}^2 & n\cdot S_3+3\cdot n\cdot(n-1)\cdot S_1S_2+n\cdot(n-1)\cdot(n-2)\cdot {S_1}^3 \\ 0 & 1 & 2\cdot n\cdot S_1 & 3\cdot n\cdot S_2+3\cdot n\cdot(n-1)\cdot {S_1}^2 \\ 0 & 0 & 1 & 3\cdot n\cdot S_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{123}$$

If, instead of seeing $I(t_1(L_N))$ as an operator similar to $L_N$, up to the previous $c$ and $s$ transformations (115), one sees $I(t_1(L_N))$ as an operator similar to $L_N$, up to transformation *c only*, one associates to $It_1$ the $4\times 4$ matrix $-M$, instead of $M$, yielding to a $(-1)^n$ factor in front of (123). If, similarly to the first Toda $L$-operator example, one just considers the iteration of $(It_1)^2$, one does not have a $(-1)^n$ factor problem, but (123) is only valid for *n even*. Note that the "time reversal" $I\cdot t_1 \rightarrow t_1\cdot I$ corresponds to the same transformation as (121), but where the function $C(x)$ is changed into the function $B(x)=1/C(x)$. When $C(x)$ is an exponential $C(x)=e^{r\cdot x}$ the entries in the matrix $M$ are not functions of $x$ but numbers. For $N=1$ and $C(x)=e^x$ one recovers the previous Toda result associated to the $2\times 2$ (shift) matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{124}$$

To sum up, transformation $\hat{K}^2$ is associated with matrix $M^2$ (with $M$ given by (122)). We are thus reduced to the analysis of the iteration of $M$ given by (122). It is clear from (123) that the orbits of $\hat{K}^2$ in the "huge" functional space $(A_0(x), A_1(x), A_2(x), A_3(x))$, correspond to a *rational curve*. This rational curve corresponds to the rational parametrization (Baxterisation) of (123) which consists in considering the integer $n$ as a real (or complex) number.

**Remark.** It is worth noticing that all these Baxterisations of $L$-operators have been performed *without using any Yang–Baxter* $(RLL=LLR)$ *hypothesis*. These calculations are similar to the calculations on the sixteen vertex model where the iteration of $\hat{K}$ yields a foliation of the whole fifteen dimensional space of the model in elliptic curves, *beyond* the Yang–Baxter integrable cases.(34)

## 8.2. Some Exercises of Baxterisation

The previous simple heuristic calculations correspond to "classical" $L$-operators. Of course all these calculations can be generalized to more general $L$-operators, in particular more "quantum" $L$-operators (see in particular (125) and (127) below). We let the reader perform a few exercises of Baxterisation.

• Show that the Baxterisation of an $R$-matrix and the Baxterisation of a monodromy matrix, built with the same $R$-matrix (like the two site monodromy matrix corresponding to the left figure (25)), yield the *same* results. For instance the *straight* Baxterisation of the two sites monodromy matrix of the sixteen vertex model *also yields elliptic curves*. This last example corresponds to matrices $A$, $B$, $C$, and $D$ in (23) and (24) being $4 \times 4$ matrices. The parameter space of the $(A, B, C, D)$-matrix (see (23) and (24)) corresponds to 32 homogeneous parameters. If the $(A, B, C, D)$-matrix corresponds to a two sites monodromy matrix of the sixteen vertex model, these 32 parameters depend, in fact, on sixteen parameters, and are thus related by quite involved algebraic relations. In practice it is almost impossible to see that an $(A, B, C, D)$-matrix actually corresponds to a two sites monodromy matrix of some "underlying" (sixteen vertex model) $R$-matrix. Fortunately the Baxterisation procedure works and provides the same results as the one for the "underlying" $R$-matrix, even if one has not been able to see that such an "underlying" $R$-matrix exists on the model.

• Let us consider the six-vertex model and its N-site monodromy matrix (see ref. 35, p. 255):

$$\mathcal{T}_N(\lambda) = \begin{bmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{bmatrix} \tag{125}$$

As a consequence of the Yang–Baxter equations, the operators $A_N(\lambda)$, $B_N(\lambda)$, $C_N(\lambda)$ and $D_N(\lambda)$ verify some quadratic relations like (see ref. 35, p. 255):

$$\begin{aligned} &B_N(\lambda) \cdot A_N(\mu) \\ &\qquad = b(\lambda-\mu) \cdot B_N(\mu) \cdot A_N(\lambda) + c(\lambda-\mu) \cdot A_N(\mu) \cdot B_N(\lambda) \\ &c(\lambda-\mu) \cdot [\, C_N(\lambda),\, B_N(\mu)] \\ &\qquad = b(\lambda-\mu) \cdot A_N(\mu) \cdot D_N(\lambda) - A_N(\lambda) \cdot D_N(\mu), \ldots \end{aligned} \tag{126}$$

where $b(\lambda-\mu) = \sin(2\eta)/\sin(\lambda-\mu+2\eta)$ and $c(\lambda-\mu) = \sin(\lambda-\mu)/\sin(\lambda-\mu+2\eta)$. Show that the action of $\hat{K}^2$ (and not $\hat{K}$) on the N-site monodromy matrix $\mathcal{T}_N(\lambda)$ amounts to a simple shift of the "spectral parameter" $\lambda$. Hint: it is easy to see that the inversion relation corresponds to $\lambda \rightarrow -\lambda$. The difficult point is to show that the action of $t_1$ on $\mathcal{T}_N(\lambda)$ is associated with: $\lambda \rightarrow 2\eta - \lambda$.

• Let us recall the $L$-operator corresponding to the Sine–Gordon model:[35]

$$L = \begin{bmatrix} u^- & 1/4m(v^-/\lambda - \lambda v^+) \\ 1/4m(\lambda v^- v^+/\lambda) & u^+ \end{bmatrix} \qquad (127)$$

where $u^\pm$ and $v^\pm$ are the Weyl operators: $u^\pm = e^{\pm i\beta/4p}$ and $v^\pm = e^{\pm i\beta/2q}$. The spectral parameter $\lambda$ is explicit here. Show that the inverse transformation $\hat{I}$, and the transformation $\hat{K}^2$, yield very simple multiplicative transformations on the spectral parameter $\lambda$. In the classical limit this $L$-operator becomes:

$$L = -i \cdot \begin{bmatrix} -2i\gamma\pi & 1/4m(\lambda e^{-1/2i\phi} - e^{1/2i\phi}/\lambda) \\ 1/4m(\lambda e^{1/2i\phi} - e^{-1/2i\phi}/\lambda) & 2i\gamma\pi \end{bmatrix} \qquad (128)$$

In this classical limit what means the inversion relation $\hat{I}$?

• Is it possible to Baxterise the $L$-operator:

$$L = \begin{bmatrix} u + y \cdot \frac{d}{dy} - l & \frac{d}{dy} \\ -y^2 \cdot \frac{d}{dy} + 2ly & u - y \cdot \frac{d}{dy} + l \end{bmatrix} \qquad (129)$$

where one recognizes[75] the realization of the finite dimensional Lie algebra $\widehat{sl}(2)$ in terms of differential operators:

$$e = \frac{d}{dy}, \qquad h = -2y \cdot \frac{d}{dy} + 2l, \qquad f = -y^2 \cdot \frac{d}{dy} + 2ly \qquad (130)$$

What is the "status" of the $u$ and $l$ parameters from a Baxterisation point of view: gauge parameters, spectral parameters...?

• Show that the Baxterisation of the $R$-matrix[76]

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 1-u & 0 \\ 0 & 0 & 1 & 0 \\ 1-u & 0 & 0 & u \end{bmatrix} \qquad (131)$$

yields a *finite order* group: the inverse $\hat{I}$ amounts to changing $u$ into $1/u$, and transformation $\hat{K}^2 = t_1 \cdot \hat{I} \cdot t_1 \cdot \hat{I}$ leaves the $R$-matrix (131) invariant: $\hat{K}^2(R) = R$.

- Show that the Baxterisation of the $R$-matrix[77]

$$R(q) = \begin{bmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & 1 & (q-1)/\sqrt{q} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{bmatrix}$$

$$\text{yields} \quad R(q, n) = \begin{bmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & 1 & (q-1)\, q^{n-1/2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{bmatrix} \tag{132}$$

Hint: the inverse of $R(q)$ is $R(1/q)$ but $\hat{K}^{2n}(R(q)) = R(q, n)$.

- Recalling that the quantum group $SL_q(2)$ is a Hopf algebra with the antipode $S$ given explicitly by:[78]

$$S(T) = S \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} D & -B/q \\ -qC & A \end{bmatrix} \tag{133}$$

show that the antipode $S$ plays exactly the role of the inverse in the Baxterisation procedure. One will use the relations:

$$\begin{aligned} &CD - qDC = 0, \qquad qBA - AB = 0 \\ &AD - DA = \left(q - \frac{1}{q}\right) \cdot BC, \qquad BC = CB \end{aligned} \tag{134}$$

Hint: calculate the product $T \cdot S(T)$ and introduce the quantum determinant $\det_q(T) = AD - qBC$. This gives:

$$T \cdot S(T) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} D & -B/q \\ -qC & A \end{bmatrix} = \det_q(T) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{135}$$

Introducing transformation $t_1$:

$$t_1(T) = t_1 \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{136}$$

let us consider the infinite dihedral group generated by these two involutions $S$ and $t_1$, or just the iteration of transformation $t_1 \cdot S$. What is the "quantum group meaning" of this infinite dihedral group?

• Let us recall the $L$-operator corresponding to the Liouville model on a lattice:[78]

$$L = \begin{bmatrix} \sqrt{1+e^{2Q+ih}}\, e^{P} & e^{Q} \\ e^{Q} & e^{-P}\, \sqrt{1+e^{2Q+ih}} \end{bmatrix} \tag{137}$$

where $P$ and $Q$ verify the Heisenberg commutation relation $[P, Q] = ih$, or the Hermann–Weyl relation $e^{P} e^{Q} = e^{ih} e^{Q} e^{P}$. Try to Baxterise it, or remark that this problem reduces to the previous Hopf algebra, with a quantum determinant equal to 1.

• Is it possible to Baxterise the universal $R$-matrix of $U_q(sl_2)$ which corresponds to the formal series:[78]

$$R = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(q-q^{-1})^k}{[k]!}\, q^{1/2H \otimes H} E^k \otimes F^k \tag{138}$$

## 9. CONCLUSION

The Baxterisation procedure is very efficient, and powerful, and is *not restricted* to any Bethe Ansatz framework, or any particular "mathematical object:" one can actually Baxterise $R$-matrices, monodromy matrices, $L$-operators, quantum Hamiltonians,... in order to see the spectral parameter, and the integrability, become crystal clear. Typically, all the calculations sketched here, can be performed for $R$ belonging to a quite general algebra $R = \sum_i c_i \cdot A_i$, the generators $A_i$ being *not necessarily associated with semi-simple Lie algebras*: they can, for instance, be elements of a *Bose–Mesner algebra*[80] associated with *distance regular graphs*...The Baxterisation procedure corresponds to very simple calculations, namely performing the inverse of a matrix, or of some element of an algebra, and performing permutations of the entries of a matrix, and combining these two transformations to get a (generically) *infinite order* (birational, polynomial) transformation one studies as a *discrete dynamical system*. The Baxterisation procedure is a very powerful tool enabling to find, or simply analyse, Yang–Baxter integrable models. It is probably the quickest, and most powerful, way to get the solution (parametrization) of the Yang–Baxter equations, *even if this parametrization is extremely complicated*. Actually, the problem of the parametrization of the Yang–Baxter equations is often a quite difficult, or subtle, one. Recalling the free-fermion condition for the asymmetric eight vertex model, it is not always clear to see if a variable is a spectral parameter, an invariant (like the modulus of elliptic functions), or simply a gauge variable[81, 82] (see also ref. 83). A variable can be seen as a spectral parameter for a row-to-row transfer matrix, and as an invariant for the column-to-column transfer matrix, looking "like a gauge variable," but corresponding, in fact, to symmetries like (34)... The Baxterisation procedure is a very efficient method to clarify these "subtleties."

It is striking to note that the Baxterisation procedure actually provides results *beyond the Yang–Baxter integrable framework*. The canonical elliptic parameterization of the sixteen vertex model is such a good example.[34] What does the *integrability of discrete symmetries* of the parameter space of the model mean *outside* the Yang–Baxter integrable framework?[84] Could it be possible that it may help to solve the model *in the absence* of Yang–Baxter integrability? Is integrability restricted to Yang–Baxter integrability? What is integrability? In our introduction we have recalled that the Yang–Baxter structure is a *sufficient* condition for the commutation of $q^N \times q^N$ transfer matrices (for any $N$), which is a fundamental property for the integrability of the model. However, if one *just* wants to calculate the partition function of the model (largest eigenvalue), *one does not need such commutation property in the whole* $q^N$ *dimensional space*: if the transfer matrix can be block diagonalized, the commutation in some block including the eigenvector corresponding to the largest eigenvalue, is *sufficient to calculate* exactly the partition function per site. The exact calculation of the partition function on the so-called "disorder solutions"[85] is such an example: the disorder solution "calculability" is *not a Yang–Baxter integrability*.[86] Other examples exist in the literature called "quasi-integrability:" they correspond to Hamiltonians for which one can *only* find exactly the ground state and the corresponding eigenvalue.

Along this line let us also recall the ideas developed by V. Jones on *planar algebras*[87] where he considers *local relations*, similar to the $ABC = CBA$ Yang–Baxter relations, that could be sufficient to calculate *global* objects like generating functions equivalent to partition functions. He introduces, for instance, *deformations* of Yang–Baxter relations, like $ABC = CBA + S$, where $S$ is "something" with "at most" two $R$ matrices (see ref. 87). In such a framework one does not have a commutation of transfer matrices anymore. One could easily imagine to Baxterise the $ABC = CBA + S$ relation in a similar way it can be done for the $ABC = CBA$ Yang–Baxter relations.[4, 5] One could thus imagine the following situation: a parameterization of the deformed Yang–Baxter relation in terms of elliptic curves, no Yang–Baxter integrability stricto sensu, but, may be, a possibility to calculate exactly the partition function per site!

Leaving these speculative "dreams" let us just underline the fact that, beyond the Yang–Baxter framework, the Baxterisation procedure has already provided a very large number of new exact results for lattice models in statistical mechanics[88, 89] and, more generally, for *discrete dynamical systems*.[90, 91]

## APPENDIX A. POLYNOMIAL REPRESENTATIONS OF $\lambda \to M \cdot \lambda$

For $N=7, 11$, one has a completely similar structure than the one depicted in Section 6, for the polynomial representation of the multiplication of the shift by $N$, namely $(J_x, J_y, J_z) \to (J_x^{(N)}, J_y^{(N)}, J_z^{(N)})$ where $J_x^{(N)} = J_x \cdot P_x^{(N)}(J_x, J_y, J_z)$ (see (56)).

For $N=7$ polynomial $P_x^{(7)}(J_x, J_y, J_z)$ reads:

$$
\begin{aligned}
P_x^{(7)}&(J_x, J_y, J_z)\\
&= 7J_y{}^{24}J_z{}^{24} - (J_z{}^2 - J_y{}^2)^{12}\, J_x{}^{24} - 28J_y{}^{22}J_z{}^{22}(J_z{}^2 + J_y{}^2)\, J_x{}^2\\
&\quad - 14J_y{}^{20}J_z{}^{20}(3J_y{}^4 + 3J_z{}^4 - 14J_z{}^2J_y{}^2)\, J_x{}^4\\
&\quad + 4J_y{}^{18}J_z{}^{18}(J_z{}^2 + J_y{}^2)(121J_y{}^4 - 250J_z{}^2J_y{}^2 + 121J_z{}^4)\, J_x{}^6\\
&\quad - 3J_y{}^{16}J_z{}^{16}(437J_z{}^4 + 726J_z{}^2J_y{}^2 + 437J_y{}^4)(J_z{}^2 - J_y{}^2)^2\, J_x{}^8\\
&\quad - 28J_z{}^6J_y{}^6(J_z{}^2 + J_y{}^2)(13J_y{}^4 + 38J_z{}^2J_y{}^2 + 13J_z{}^4)(J_z^2 - J^2{}_y)^6\, J_x{}^{18}\\
&\quad + 24J_y{}^{14}J_z{}^{14}(J_z{}^2 + J_y{}^2)(75J_y{}^4 + 106J_z{}^2J_y{}^2 + 75J_z{}^4)(J_z{}^2 - J_y{}^2)^2\, J_x{}^{10}\\
&\quad + 2J_z{}^4J_y{}^4(59J_z{}^8 + 332J_z{}^6J_y{}^2 + 626J_z{}^4J_y{}^4\\
&\quad + 332J_z{}^2J_y{}^6 + 59J_y{}^8)(J_z{}^2 - J_y{}^2)^6\, J_x{}^{20}\\
&\quad - 4J_z{}^2J_y{}^2(J_y{}^2 + 3J_z{}^2)(3J_y{}^2 + J_z{}^2)(J_z{}^2 + J_y{}^2)\\
&\quad \times (J_y{}^4 + 6J_z{}^2J_y{}^2 + J_z{}^4)(J_z^2 - J_y^2)^6\, J_x{}^{22}\\
&\quad - 12J_y{}^{12}J_z{}^{12}(105J_z{}^8 + 420J_z{}^6J_y{}^2 + 422J_z{}^4J_y{}^4\\
&\quad + 420J_z{}^2J_y{}^6 + 105J_y{}^8)(J_z^2 - J_y^2)^2\, J_x{}^{12}\\
&\quad + 8J_z{}^{10}J_y{}^{10}(J_z{}^2 + J_y{}^2)(21J_y{}^8 + 420J_z{}^2J_y{}^6\\
&\quad - 50J_z{}^4J_y{}^4 + 420J_z{}^6J_y{}^2 + 21J_z{}^8)(J_z^2 - J_y^2)^2\, J_x{}^{14}\\
&\quad + J_z{}^8J_y{}^8(7J_y{}^4 + 7J_z{}^4 - 30J_z{}^2J_y{}^2)(63J_y{}^8\\
&\quad + 84J_z{}^2J_y{}^6 - 38J_z{}^4J_y{}^4 + 84J_z{}^6J_y{}^2 + 63J_z{}^8)(J_z^2 - J_y^2)^2\, J_x{}^{16}
\end{aligned}
\tag{139}
$$

The modulus (51) is invariant by the polynomial transformation $(J_x, J_y, J_z) \to (J_x^{(7)}, J_y^{(7)}, J_z^{(7)})$, corresponding to the multiplication of the shift by seven.

The multiplication of the shift by eleven has a polynomial representation $(J_x, J_y, J_z) \to (J_x^{(11)}, J_y^{(11)}, J_z^{(11)})$ satisfying (56) for $N=11$, where $P_x^{(11)}(J_x, J_y, J_z)$ is a homogeneous polynomial of degree 120 sum of 496 monomial expressions:

$$
\begin{aligned}
P_x^{(11)}(J_x, J_y, J_z) = & -30045015J_y^{40}J_z^{20}J_x^{60} + 2035800J_y^{46}J_z^{14}J_x^{60} \\
& + 142506J_y^{50}J_z^{10}J_x^{60} - 27405J_y^{8}J_z^{52}J_x^{60} \\
& + 54627300J_y^{38}J_z^{22}J_x^{60} - 435J_z^{56}J_y^{4}J_x^{60} \\
& + 54627300J_y^{22}J_z^{38}J_x^{60} + \cdots
\end{aligned} \tag{140}
$$

The actual expression of $P_x^{(11)}(J_x, J_y, J_z)$ will be given elsewhere.

When $N$ is not a prime number one has slightly modified results: one does not have (56) anymore. Actually, the polynomial representation of the multiplication of the shift by six can be obtained in many different ways, namely substituting $J^{(2)}$ in $J^{(3)}$, or $J^{(3)}$ in $J^{(2)}$, or by various eliminations between various biquadratic curves $\Gamma_i$. The result reads $(J_x, J_y, J_z) \to (J_x^{(6)}, J_y^{(6)}, J_z^{(6)})$ where:

$$
\begin{aligned}
J_x^{(6)} &= J_x^{(2)} \cdot \tilde{P}_x^{(6)}(J_x, J_y, J_z) \\
J_y^{(6)} &= J_y^{(2)} \cdot \tilde{P}_y^{(6)}(J_x, J_y, J_z) = J_y^{(2)} \cdot \tilde{P}_x^{(6)}(J_y, J_z, J_x) \\
J_z^{(6)} &= J_z^{(2)} \cdot \tilde{P}_z^{(6)}(J_x, J_y, J_z) = J_z^{(2)} \cdot \tilde{P}_x^{(6)}(J_z, J_x, J_y)
\end{aligned} \tag{141}
$$

such that $J_y^{(6)}(J_x, J_y, J_z) = J_x^{(6)}(J_y, J_z, J_x)$ and $J_z^{(6)}(J_x, J_y, J_z) = J_x^{(6)}(J_z, J_x, J_y)$. The $\tilde{P}^{(6)}$'s and the $J^{(6)}$'s are functions of $J_x^2$, $J_y^2$ and $J_z^2$. One also has $J_x^{(6)}(J_x, J_y, J_z) = J_x^{(6)}(J_x, J_z, J_y)$, $J_y^{(6)}(J_x, J_y, J_z) = J_y^{(6)}(J_z, J_y, J_x)$, $J_z^{(6)}(J_x, J_y, J_z) = J_z^{(6)}(J_y, J_x, J_z)$, and $\tilde{P}_x^{(6)}(J_x, J_y, J_z) = \tilde{P}_x^{(6)}(J_x, J_z, J_y)$, $\tilde{P}_y^{(6)}(J_x, J_y, J_z) = \tilde{P}_y^{(6)}(J_z, J_y, J_x)$, $\tilde{P}_z^{(6)}(J_x, J_y, J_z) = \tilde{P}_z^{(6)}(J_y, J_x, J_z)$.

All these expressions can thus be deduced from $\tilde{P}_x^{(6)}(J_x, J_y, J_z)$:

$$
\begin{aligned}
&\tilde{P}_x^{(6)}(J_x, J_y, J_z) \\
&\quad = {J_y}^{16}{J_z}^{16} - 8{J_y}^{14}{J_z}^{14}({J_z}^2 + {J_y}^2)\,{J_x}^2 \\
&\qquad + 4{J_y}^{12}{J_z}^{12}(-10{J_z}^2{J_y}^2 + 7{J_y}^4 + 7{J_z}^4)\,{J_x}^4 \\
&\qquad - 56{J_z}^{10}{J_y}^{10}({J_z}^2 + Jy^2)(J_z - J_y)^2\,(J_z + J_y)^2\,{J_x}^6 \\
&\qquad + 2{J_z}^8{J_y}^8(35{J_z}^4 + 114{J_z}^2{J_y}^2 + 35{J_y}^4)(J_z - J_y)^2\,(J_z + J_y)^2\,{J_x}^8 \\
&\qquad - 8{J_z}^6{J_y}^6({J_z}^2 + {J_y}^2)(7Jy^4 + 18{J_z}^2{J_y}^2 + 7{J_z}^4)(J_z - J_y)^2\,(J_z + J_y)^2\,{J_x}^{10} \\
&\qquad + 4{J_z}^4{J_y}^4(7{J_y}^4 + {J_z}^4)({J_y}^4 + 7{J_z}^4)(J_z - J_y)^2\,(J_z + J_y)^2\,{J_x}^{12} \\
&\qquad - 8{J_z}^2{J_y}^2({J_z}^2 + {J_y}^2)(J_z - J_y)^6\,(J_z + J_y)^6\,{J_x}^{14} \\
&\qquad + ({J_z}^4 + 14{J_z}^2{J_y}^2 + {J_y}^4)(J_z - J_y)^6\,(J_z + J_y)^6\,{J_x}^{16}
\end{aligned} \tag{142}
$$

Note that $J_x^{(6)}$ is a homogeneous polynomial expression of degree 36.

Similarly, the polynomial representation of the multiplication of the shift by nine reads $(J_x, J_y, J_z) \rightarrow (J_x^{(9)}, J_y^{(9)}, J_z^{(9)})$ where:

$$\begin{aligned} J_x^{(9)} &= J_x^{(3)} \cdot \tilde{P}_x^{(9)}(J_x, J_y, J_z) \\ J_y^{(9)} &= J_y^{(3)} \cdot \tilde{P}_y^{(9)}(J_x, J_y, J_z) = J_y^{(3)} \cdot P_x^{(9)}(J_y, J_z, J_x) \\ J_z^{(9)} &= J_z^{(3)} \cdot \tilde{P}_z^{(9)}(J_x, J_y, J_z) = J_z^{(3)} \cdot \tilde{P}_x^{(9)}(J_z, J_x, J_y) \end{aligned} \tag{143}$$

The expression of $\tilde{P}_x^{(9)}(J_x, J_y, J_z)$ will be given elsewhere. It is, again, a function of $J_x^2$, $J_y^2$ and $J_z^2$. It is the sum of 190 monomial expressions of degree 72, $J_x^{(9)}$ being a homogeneous polynomial expression of degree 81.

## APPENDIX B. FINITE ORDER CONDITIONS

It can be shown that the points of the Baxter model on the algebraic varieties:

$$V^{(3)}(J_x, J_y, J_z) = J_x J_y + J_z J_x + J_z J_y$$
$$\hat{V}_z^{(3)}(J_x, J_y, J_z) = -J_x J_y + J_z J_x + J_z J_y$$

are actually such that $K^6(R) = \zeta \cdot R$. One has the following factorization property:

$$\begin{aligned} &V^{(3)}(J_x^{(2)}, J_y^{(2)}, J_z^{(2)}) \\ &\quad = V^{(3)}(J_x, J_y, J_z) \cdot \hat{V}_x^{(3)}(J_x, J_y, J_z) \cdot \hat{V}_y^{(3)}(J_x, J_y, J_z) \cdot \hat{V}_z^{(3)}(J_x, J_y, J_z) = 0 \end{aligned} \tag{144}$$

where $\hat{V}_x^{(3)}(J_x, J_y, J_z) = \hat{V}_z^{(3)}(J_y, J_z, J_x)$ and $\hat{V}_y^{(3)}(J_x, J_y, J_z) = \hat{V}_z^{(3)}(J_z, J_x, J_y)$. One has the relation:

$$\hat{V}_z^{(3)}(J_x^{(2)}, J_y^{(2)}, J_z^{(2)}) - P_z^{(3)}(J_x, J_y, J_z) = 0 \tag{145}$$

Note that the points of the Baxter model on the algebraic varieties $P_x^{(3)}(J_x, J_y, J_z) = 0$, or $P_y^{(3)}(J_x, J_y, J_z) = 0$, or $P_z^{(3)}(J_x, J_y, J_z) = 0$, are such that $K^{12}(R) = \zeta \cdot R$. Relation (145) is in agreement with the fact that $(J_x, J_y, J_z) \rightarrow (J_x^{(2)}, J_y^{(2)}, J_z^{(2)})$ is a representation of the shift doubling. The points of order six (namely $K^6(R) = \zeta \cdot R$) correspond to (144), their image by the shift doubling $(J_x, J_y, J_z) \rightarrow (J_x^{(2)}, J_y^{(2)}, J_z^{(2)})$ giving $P_x^{(3)}(J_x, J_y, J_z) \cdot P_y^{(3)}(J_x, J_y, J_z) \cdot P_z^{(3)}(J_x, J_y, J_z) = 0$, together, of course, with (144).

Let us note that the points of the algebraic variety $\hat{V}_z^{(3)}(J_x, J_y, J_z) = 0$ are (projectively) of order *three* for the Baxter model: $K^3(R) = \zeta \cdot R$. The points of the Baxter model on the variety:

$$\begin{aligned} V_y^{(5)}(J_x, J_y, J_z) = & J_z^2 J_x J_y^3 - 2J_z^2 J_x^2 J_y^2 + J_z^2 J_x^3 J_y + J_z J_x^2 J_y^3 \\ & - J_z J_x^3 J_y^2 - J_x^3 J_y^3 - J_z^3 J_x J_y^2 \\ & + J_z^3 J_x^2 J_y + J_z^3 J_x^3 - J_x^3 J_y^3 = 0 \end{aligned} \tag{146}$$

are of order five $K^5(R) = \zeta \cdot R$. The points of the Baxter model on the algebraic variety:

$$\begin{aligned} \hat{V}_y^{(5)}(J_x, J_y, J_z) = V_y^{(5)}(J_y, J_x, J_z) = & J_z^2 J_x^3 J_y - 2J_z^2 J_x^2 J_y^2 + J_z^2 J_x J_y^3 + J_z J_x^3 J_y^2 \\ & - J_z J_x^2 J_y^3 - J_z^3 J_x^3 - J_z^3 J_x^2 J_y + J_z^3 J_x J_y^2 + J_z^3 J_y^3 - J_x^3 J_y^3 = 0 \end{aligned} \tag{147}$$

are of order ten: $K^{10}(R) = \zeta \cdot R$. The point of the Baxter model on the variety $P_x^{(5)}(J_x, J_y, J_z) = 0$, $P_y^{(5)}(J_x, J_y, J_z) = 0$, or $P_z^{(5)}(J_x, J_y, J_z) = 0$, are of order *twenty*: $K^{20}(R) = \zeta \cdot R$. Let us note that:

$$\begin{aligned} V_y^{(5)}(J_x^{(2)}, J_y^{(2)}, J_z^{(2)}) + P_y^{(5)}(J_x, J_y, J_z) = 0 \\ \hat{V}_y^{(5)}(J_x^{(2)}, J_y^{(2)}, J_z^{(2)}) + P_x^{(5)}(J_x, J_y, J_z) = 0 \end{aligned} \tag{148}$$

which is in agreement with the fact that $(J_x, J_y, J_z) \to (J_x^{(2)}, J_y^{(2)}, J_z^{(2)})$ represents the shift doubling.

**Remark.** Relation (148) is in agreement with the shift doubling, however one seems to have an apparent contradiction with previous relations. The points of (146), namely $V_y^{(5)}(J_x, J_y, J_z) = 0$ being of order five, one expects that their image by the shift doubling will give points of order ten, like (147), and not points of order twenty, like $P_y^{(5)}(J_x, J_y, J_z) = 0$. In fact, this algebraic variety is an order five algebraic variety *only when restricted to the Baxter model*. For the sixteen vertex model one can actually show that $V_y^{(5)}(J_x, J_y, J_z) = 0$ and $\hat{V}_y^{(5)}(J_x, J_y, J_z) = 0$, are (codimension-one) algebraic varieties of order *ten*, the algebraic varieties of order five being higher codimension algebraic varieties.

## ACKNOWLEDGMENTS

One of us (JMM) would like to thank R. J. Baxter for so many discussions and for so many warm and friendly encouragements, in the last two decades. One of us (JMM) would also like to thank P. A. Pearce for his great hospitality in the Mathematics and Statistics Dept. of Melbourne University where part of this work has been completed. The authors would like to thank J.-C. Anglès d'Auriac for many discussions on the Baxterisation of quantum Hamiltonians.

## REFERENCES

1. P. Lochak and J. M. Maillard, *J. Math. Phys.* **27**:593 (1986).
2. J. M. Maillard, *J. Math. Phys.* **27**:2776 (1986).
3. E. Bertini, *Life and Work of L. Cremona*, Proceedings of the London Mathematical Society, Vol. 2, 1, V-XVIII.
4. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, *Phys. Lett. A* **157**:343 (1991).
5. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, *Phys. Lett. B* **260**:87 (1991).
6. J.-M. Maillard, and C.-M. Viallet, *Phys. Lett. B* **381**:269 (1996).
7. G. E. Andrews, R. J. Baxter, and P. J. Forrester, *J. Stat. Phys.* **35**:193 (1984).
8. C. N. Yang and C. P. Yang, *Phys. Rev.* **150**:321–339 (1966).
9. C. N. Yang and C. P. Yang, *Phys. Rev.* **151**:258–264 (1966).
10. M. Takahashi and M. Suzuki, *Prog. Theor. Phys.* **48**:2187–2209 (1972).
11. D. Hansel and J. M. Maillard, *Phys. Lett. A* **133**:11 (1988).
12. A. B. Zamolodchikov, *Comm. Math. Phys.* **79**:489 (1981).
13. R. J. Baxter, *Comm. Math. Phys.* **88**:185 (1983).
14. R. J. Baxter, *Physica D* **18**:321 (1986).
15. S. Boukraa, J.-M. Maillard, and G. Rollet, *J. Stat. Phys.* **78**:1195 (1995).
16. S. Boukraa and J.-M. Maillard, *Physica A* **220**:403–470 (1995).
17. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, *Phys. Rev. Lett.* **67**:1373 (1991).
18. S. Boukraa, J.-M. Maillard, and G. Rollet, *Int. J. Mod. Phys. B* **8**:2157 (1994).
19. S. Boukraa, J.-M. Maillard, and G. Rollet, *Physica A* **208**:115 (1994).
20. S. Boukraa, J.-M. Maillard, and G. Rollet, *Int. J. Mod. Phys. B* **8**:137 (1994).
21. Y. G. Stroganov, *Phys. Lett. A* **74**:116 (1979).
22. R. J. Baxter, *J. Stat. Phys.* **28**:1 (1982).
23. J. Hietarinta, *J. Phys. A* **26**:L9 (1993).
24. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A* **128**:138 (1988).
25. N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, and J.-M. Maillard, *Physica A* **264**:264 (1999).
26. N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, and J.-M. Maillard, *Phys. Lett. A* **262**:44 (1999).
27. N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, and J.-M. Maillard, *Physica D* **130**:27 (1999).
28. J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics (Springer-Verlag, 1992).
29. J. E. Sacco and F. Y. Wu, *J. Phys. A* **8**(11):1780 (1975).
30. S. M. Sergeev, V. V. Mangazeev, and Yu. G. Stroganov, *The Vertex Formulation of the Bazhanov–Baxter Model* (Australian National University, 1995). RSPhysSE Preprint.
31. I. G. Korepanov, *Tetrahedron Equation and the Algebraic Geometry*, hep-th/9401976.
32. J. Hietarinta, *J. Phys. A* **27**:5727 (1994).
33. E. H. Lieb and F. Y. Wu, Two dimensional ferroelectric models, in *Phase Transitions and Critical Phenomena*, Vol. 1, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972), pp. 331–490.
34. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, *Phys. Lett. B* **281**:315 (1992).
35. L. D. Faddeev, *40 Years in Mathematical Physics*, World Scientific Series in 20th Century Mathematics (1995), Vol. 2.
36. R. J. Baxter, *Ann. Phys.* **76**:25 (1973).

37. F. G. Frobenius, *Über das Additions-theorem der Thetafunctionen mehrerer variabeln*, Journ. für die reine und angewandte Mathematik, Vol. 89 (1880), p. 185.
38. M. Gaudin, *La Fonction d'Onde de Bethe*, Collection du C.E.A. Série Scientifique (Masson, Paris, 1983).
39. A. Zamolodchikov and Al. Zamolodchikov, *Phys. Lett. A* **91**:157 (1982).
40. R. E. Schrock, *Ann. Phys.* **120**:253 (1979).
41. F. J. Wegner, *Physica A* **68**:570 (1973).
42. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (London Academic Press, 1981).
43. D. Hilbert, *Mathem. Ann.* **36**:473 (1890).
44. A. Gaaff and J. Hijmans, *Physica A* **80**:149 (1975).
45. A. Gaaff and J. Hijmans, *Physica A* **83**:301 (1976).
46. A. Gaaff and J. Hijmans, *Physica A* **83**:317 (1976).
47. A. Gaaff and J. Hijmans, *Physica A* **94**:192 (1978).
48. A. Gaaff and J. Hijmans, *Physica A* **97**:244 (1979).
49. J. Hijmans, *Physica A* **130**:57–87 (1985).
50. J. Hijmans and H. M. Schram, *Physica A* **121**:479–512 (1983).
51. J. Hijmans and H. M. Schram, *Physica A* **125**:25 (1984).
52. H. M. Schram and J. Hijmans, *Physica A* **125**:58 (1984).
53. J. H. P. Colpa, *Physica A* **125**:425–442 (1984).
54. S. Boukraa and J.-M. Maillard, *Journal de Physique I* **3**:239 (1993).
55. R. Z. Bariev, A. Klümper, A. Schadschneider, and J. Zittartz, *Zeitschrift für Physik B* **96**:395 (1995).
56. F. H. L. Essler and V. E. Korepin, *Phys. Rev. B* **46**:9147 (1992).
57. R. Z. Bariev, Exact solution of generalized t-J models in one dimension, *J. Phys. A* **27**:3381 (1994).
58. E. Olmedilla, M. Wadati, and Y. Akutsu, *J. Phys. Soc. Japan* **56**:2298 (1987).
59. F. H. L. Essler, V. E. Korepin, and K. Shoutens, *Phys. Rev. Lett.* **68**:2960 (1992).
60. P. W. Kasteleyn, Exactly solvable lattice models, in *Proc. of the 1974 Wageningen Summer School: Fundamental Problems in Statistical Mechanics III* (North-Holland, Amsterdam, 1974).
61. H. Meyer, J. C. Anglès d'Auriac, and J.-M. Maillard, *Phys. Rev. E* **55**:5380 (1997).
62. R. J. Baxter, *Ann. Phys.* **70**:323 (1972).
63. S. Boukraa, S. Hassani, and J.-M. Maillard, *Physica A* **240**:586 (1997).
64. P. Schlottmann, *Phys. Rev. B* **36**:5177 (1987).
65. C. L. Schultz, *Phys. Rev. Lett.* **46**:629 (1981).
66. J. H. H. Perk and C. L. Schultz, *Phys. Lett. A* **84**:407 (1981).
67. M. Toda, *J. Phys. Soc. Japan* **22**:431 (1967).
68. M. Toda, *Theory of Nonlinear Lattices*, Springer series in Solid State Sciences **20** (1981).
69. S. V. Manakov, *J.E.T.P.* **65**:1392 (1973).
70. H. Flashka, *Progr. Theor. Phys.* **51**:703 (1974).
71. P. P. Kulish and E. K. Sklyanin, Quantum spectral transform method. Recent developments, in *Integrable Quantum Field Theories*, Lecture Notes in Physics, Vol. 51, pp. 61–119 (1982) (Proceedings Tvärminne, Finland, 1981).
72. V. B. Kuznetsov, M. Salerno, and E. K. Sklyanin, *Quantum Bäcklund Transformation for the Integrable DST Model*, solv-int/9908002 Submitted to J. Phys. A.
73. L. A. Takhtadzhan and L. D. Faddeev, *Russian Math. Surveys* **34**:11–68 (1979).
74. L. D. Faddeev, Integrable Models in $(1+1)$-dimensional quantum field Theory, Les Houches, Session XXXIX (1982), in *Recent advances in Field Theory and Statistical Mechanics*, J-B. Zuber and R. Stora, eds. (Elsevier Science Publishers B.V., 1984).

75. P. Bouwknegt, J. McCarthy, and K. Pilch, *Comm. Math. Phys.* **131**:125 (1990).
76. M. Ruiz-Altaba, *Phys. Lett. B* **279**:326 (1992).
77. N. Reshetikhin and F. Smirnov, *Comm. Math. Phys.* **131**:157 (1990).
78. O. Babelon, P. Cartier, and Y. Kosmann-Schwarzbach, eds., *Lectures on Integrable Systems*, in memory of Jean-Louis Verdier Proceedings of the CIMPA School World Scientific (1994), ISBN 981-02-1757-9.
79. G. Felder and C. Wieczerkowski, *Comm. Math. Phys.* **138**:583 (1991).
80. F. Jaeger, M. Matsumoto, and K. Nomura, *Bose–Mesner Algebras Related to Type II Matrices and Spin Models*, RIMS preprint (1995).
81. I. M. Krichever, *Funct. Anal. and its Appl.* **15**:92 (1981).
82. V. V. Bazhanov and Y. G. Stroganov, *Theor. Math. Phys.* **62**:253–260, **63**:519–527, **63**:604–611 (1985).
83. M. Shiroishi and M. Wadati, *J. Phys. Soc. Japan.* **64**:2795 (1995).
84. H. Meyer, J.-C. Anglès d'Auriac, J.-M. Maillard, and G. Rollet, *Physica A* **208**:223 (1994).
85. M. T. Jaekel and J. M. Mailland, *J. Phys. A* **18**:1229 (1985).
86. M. T. Jaekel and J. M. Maillard, *J. Phys. A* **17**:2079 (1984).
87. V. F. R. Jones, *Planar Algebras, I.*, math/9909027.
88. D. Hansel, J. M. Maillard, J. Oitmaa, and M. Velgakis, *J. Stat. Phys.* **48**:69 (1987).
89. J.-M. Mailland and G. Rollet, *J. Phys A* **27**:6963 (1994).
90. N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, and J.-M. Maillard, *The European Physical Journal B* **5**:647 (1998).
91. N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, and J.-M. Mailland, From Yang–Baxter equations to dynamical zeta functions for birational transformations, in *Statistical Mechanics at the Eve of the 21st Century. In Honour of J. B. McGuire on the Occasion of His 65th Birthday*, M. T. Batchelor and L. T. Wille, eds., Series on Advances in Statistical Mechanics, Vol. 14 (World Scientific, 1999), pp. 436–490.